

2024/TDC (CBCS)/EVEN/SEM/
MTMHCC-403T/234

TDC (CBCS) Even Semester Exam., 2024

MATHEMATICS

(4th Semester)

Course No. : MTMHCC-403T

(Ring Theory)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

UNIT—I

1. Answer any *two* of the following questions : $2 \times 2 = 4$

(a) Give an example to show that union of two subrings may not be a subring.

(b) Prove that in an integral domain R (with unity) the only idempotents are the zero and unity.

(c) Define characteristic of a ring. Give an example of a ring whose characteristic is 6(six).

24J/837

(Turn Over)



2. Answer either (a) and (b) or (c) and (d) :

- (a) Prove that a non-zero finite integral domain is a field. 5
- (b) If R is a division ring, then show that the centre $Z(R)$ of R is a field. 5
- (c) If D is an integral domain and if $na = 0$ for some $0 \neq a \in D$ and some integer, $n \neq 0$, then show that the characteristic of D is finite. 4
- (d) Let R be a commutative ring with unity. Show that—
- (i) if $a \in R$ is a unit, then a is not nilpotent;
- (ii) if $n \in R$ is nilpotent, then $1+n$ is a unit;
- (iii) the sum of a nilpotent element and a unit is a unit. 6

UNIT—II

3. Answer any two questions : $2 \times 2 = 4$

- (a) Prove that $6\mathbb{Z}$ is an ideal of \mathbb{Z} .
- (b) Define prime ideal. Give an example of it.
- (c) Give an example of an ideal of a ring which is neither prime nor maximal with justification.

24J/837

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4. Answer either (a) and (b) or (c) and (d) :

- (a) Let R be a ring with unity, such that R has no right ideals except $\{0\}$ and R , then show that R is a division ring. 5
- (b) Let R be a commutative ring with unity. Show that every maximal ideal of R is prime ideal. 5
- (c) Let $A \neq R$ be an ideal of R , then for any $x \in R$, $x \notin A$, if $A + (x) = R$, show that A is maximal ideal of R and conversely. 5
- (d) Show that a commutative ring R is an integral domain if and only if $\{0\}$ is a prime ideal. 5

UNIT—III

5. Answer any two questions : $2 \times 2 = 4$

- (a) If $f: R \rightarrow R'$ be a ring homomorphism then prove that—
- (i) $f(0) = 0'$
- (ii) $f(-a) = -f(a)$
- where $0, 0'$ are zeros of the rings R and R' respectively.

24J/837

(Turn Over)

- (b) State fundamental theorem of ring homomorphism.
- (c) Let I be an ideal of a ring R . Show that
- if R is commutative then so is R/I
 - if R has unity 1 then $1 + I$ is unity of R/I .

6. Answer either (a) and (b) or (c) and (d) :

- (a) Let \mathbb{Z} be the ring of integers. Show that the only homomorphisms from $\mathbb{Z} \rightarrow \mathbb{Z}$ are the identity and zero mapping. 5
- (b) Let $B \subseteq A$ be two ideals of a ring R , then prove that $\frac{R}{A} \cong \frac{R/B}{A/B}$. 5
- (c) Prove that any homomorphism of a field is either a monomorphism or takes each element to zero. 5
- (d) Find all the ring homomorphisms from $\mathbb{Z}_{20} \rightarrow \mathbb{Z}_{30}$. 5

UNIT—IV

7. Answer any two of the following questions : $2 \times 2 = 4$

- (a) Let $R[x]$ be the ring of polynomials over a ring R . Then prove that if R has unity, then $R[x]$ has unity.

24J/837

(Continued)

- (b) Give an example to show that quotient ring of an integral domain may not be an integral domain.
- (c) Define Euclidean domain. Give an example of it.

8. Answer either (a) and (b) or (c) and (d) :

- (a) Let $R[x]$ be the ring of polynomials of a ring R . Then prove that R is an integral domain if and only if $R[x]$ is an integral domain. 5
- (b) Let a, b be two non-zero elements of a Euclidean domain R . If b is not a unit in R , then prove that $d(a) < d(ab)$. 5
- (c) If F is a field then prove that $F[x]$ is a Euclidean domain. 6
- (d) Show that in a principal ideal domain, every non-zero prime ideal is maximal. 4

UNIT—V

9. Answer any two of the following questions : $2 \times 2 = 4$

- (a) Give an example of a UFD which is not a PID.
- (b) State Eisenstein's criterion.
- (c) Show that $\frac{Q[x]}{I}$, where $I = \langle x^2 - 5x + 6 \rangle$, is not a field.

24J/837

(Turn Over)



10. Answer either (a) and (b) or (c) and (d) :

(a) Prove that an integral domain R with unity is a UFD if and only if every non-zero, non-unit element is a finite product of primes. 6

(b) If R is an integral domain with unity and a is an irreducible element of R , then prove that a is irreducible element of $R[x]$. 4

(c) If F is a field, then prove that an ideal $\langle P(x) \rangle \neq \{0\}$ in $F[x]$ is maximal if and only if $P(x)$ is irreducible in $F[x]$. 4

(d) If R is a UFD, then prove that $R[x]$ is also a UFD. 6

2024/TDC (CBCS)/EVEN/SEM/
MTMHCC-403T/234

24J-360/837