2024/TDC (CBCS)/EVEN/SEM/
MTMHCC–402T/233

TDC (CBCS) Even Semester Exam., 2024

MATHEMATICS

(4th Semester)

Course No. : MTMHCC–402T

(Riemann Integration and Series of Functions)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following questions : $2\times2=4$

(a) Write two partitions of the interval [0, 1] such that one of them is a refinement of the other.

(b) Give example with justification of a bounded function on [0, 2024] which is not Darboux integrable.

(c) Define Riemann sum of a function with respect to a tagged partition of the domain $[a, b]$. Hence, define Riemann integral.

2. Answer *either* [(a) and (b)] *or* [(c) and (d)] : 10

(a) Show that a bounded function f on $[a, b]$ is Darboux integrable iff for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$. 5

(b) Show that Riemann integral of a function, if it exists, is unique. 5

(c) If P and P^* are two partitions of $[a, b]$ where P^* is a refinement of $\mathbb{P}$, show that

$$U(P^*, f) - L(P^*, f) < U(P, f) - L(P, f)$$ 5

(d) Show that every monotone function on $[a, b]$ is Riemann integrable. 5

UNIT—II

3. Answer any *two* of the following questions : 2×2=4

(a) If $f(x) \geq 0 \; \forall x \in [a, b]$, show that

$$\int_a^b f\,dx \geq 0$$

(b) If f is integrable on $[a, b]$, and g is not integrable on $[a, b]$, what can you conclude about the integrability of $f + g$ on $[a, b]$? Justify your answer.

(c) Give example with justification, of a function f such that $|f|$ is integrable but f is not integrable.

4. Answer *either* [(a) and (b)] *or* [(c) and (d)] : 10

(a) Show that the sum of two integrable functions is integrable. 5

(b) Let f be bounded and integrable on $[a, b]$ and there exists a function F on $[a, b]$ such that $F' = f$ on $[a, b]$, then show that

$$\int_a^b f\,dx = F(b) - F(a)$$ 5

(c) Show that if f is integrable on $[a, b]$, then it is also integrable on $[a, c]$ and $[c, b]$ where $c \in (a, b)$. Also show that

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$$ 5

(d) If f and g are integrable on $[a, b]$ and g keeps the same sign over $[a, b]$, then prove that there exists a number μ lying between the bounds of f such that

$$\int_a^b fg\,dx = \mu \int_a^b g\,dx$$ 5

UNIT—III

5. Answer any *two* of the following questions : 2×2=4

(a) Test the convergence of

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

(b) Write the comparison test for testing the convergence of $\int_a^b f\,dx$ at a, in limit form.

(c) Evaluate :

$$\int_0^1 x^4(1-x)^5\,dx$$

6. Answer *either* [(a) and (b)] *or* [(c) and (d)] : 10

(a) Examine the convergence of

$$\int_{-1}^1 \frac{dx}{(2-x)\sqrt{1-x^2}}$$

and evaluate the value, in case it converges. 5

(b) Show that

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

converges iff $m, n > 0$. 5

(c) Find the values of p for which

$$\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$$

converges. 4

(d) Prove that

$$\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \beta(m, n)$$

and hence show that

$$\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \beta(m, n)$$ 6

UNIT—IV

7. Answer any *two* of the following questions : 2×2=4

(a) Define uniform convergence of a sequence of functions on the closed interval $[a, b]$. Illustrate with an example.

(b) Check whether the sequence of functions $< x^n >$ on $[0, 1]$ converges uniformly.

(c) Use Weirstrass M-test to check the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$$

8. Answer *either* [(*a*) and (*b*)] *or* [(*c*) and (*d*)] : 10

(*a*) Show that the sequence of functions

$$\left\langle \frac{x}{x+n} \right\rangle$$

converges uniformly in $[0, a]$ for any $a > 0$ but only pointwise in $[0, \infty)$. 5

(*b*) State and prove the Weirstrass M-test for uniform convergence of a series of functions. 5

(*c*) State and prove the Cauchy's criterion for uniform convergence of a sequence of functions. 5

(*d*) Test the uniform convergence of—

(*i*) $\sum \frac{\sin(x^2 + n^2 x)}{n(n+1)}$;

(*ii*) $\sum \frac{\cos nx}{x^p}$ 5

UNIT—V

9. Answer any *two* of the following questions : 2×2=4

(*a*) Find the radius of convergence of the power series

$$1 + x + x^2 + x^3 + \cdots$$

(*b*) Give example of a power series that is convergent only at $x = 0$ and of another power series that converges for all $x \in \mathbb{R}$.

(*c*) Given that a power series converges for $x = 2$, find an interval in which it is absolutely convergent. Also, find a lower bound for its radius of convergence.

10. Answer *either* [(*a*) and (*b*)] *or* [(*c*) and (*d*)] : 10

(*a*) If a power series $\sum a^n x^n$ converges for $x = r$, then prove that it converges absolutely for every x with $|x| < |r|$. Give example to show that the power series may not be absolutely convergent at $x = r$. 5

(*b*) Determine the radius of convergence and the exact interval of convergence of the following power series : 3+2=5

(*i*) $\sum \frac{(n+1)x^n}{(n+2)(n+3)}$

(*ii*) $\sum \frac{2^n x^n}{n!}$

(c) Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^\alpha}$$

for various values of α. 5

(d) If a power series $\sum a^n x^n$ has radius of convergence R, prove that for any positive integer m the series $\sum a_n x^{mn}$ has radius of convergence $R^{1/m}$. 5

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