

**2024/TDC (CBCS)/EVEN/SEM/
MTMHCC-602T/238**

TDC (CBCS) Even Semester Exam., 2024

MATHEMATICS

(6th Semester)

Course No. : MTMHCC-602T

(Linear Algebra)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

UNIT—I

1. Answer any two questions : 2×2=4

(a) Show that the set $L = \{(x, y) \mid 2x + 3y = 0\}$ is a subspace of $\mathbb{R}^2(\mathbb{R})$.

(b) Check if the vectors $(1, 0, 0)$, $(1, 0, 1)$ and $(1, 1, 1)$ are linearly independent.

(c) If $\{v_1, v_2, \dots, v_k\}$ are linearly dependent vectors in a vector space $V(\mathbb{R})$, show that one of the vectors is a linear combination of the remaining vectors.

2. Answer either [(a) and (b)] or [(c) and (d)] :

- (a) Show that the intersection of finite number of subspaces of a vector space is also a subspace of the vector space. Give example to show that the same is not true for union. 4
- (b) Let V be a finite dimensional vector space. Show that any two bases of V have the same number of elements. 6
- (c) If W_1 and W_2 are subspaces of finite dimensional vector, show that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$
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- (d) Find a basis and the dimension of each of the following subspaces of \mathbb{R}^n :
 $V = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 + x_2 + \dots + x_n = 0\}$
 $W = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 = x_2 = x_3\}$
 where $n \geq 3$. 5

UNIT—II

3. Answer any two questions : $2 \times 2 = 4$

- (a) Define linear transformation from a vector space to another. Give an example, with justification, of a linear transformation from \mathbb{R}^3 to \mathbb{R}^4 .

(b) Justify True or False :

If S is a linearly independent set in a vector space V and $T: V \rightarrow V$ is a linear transformation, then $T(S)$ is also linearly independent.

- (c) If $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is such that $T(1, 0) = (2, 3)$, $T(0, 2) = (3, 2)$, find $T(x, y)$ for any $(x, y) \in \mathbb{R}^2$.

4. Answer either [(a) and (b)] or [(c) and (d)] :

- (a) If $T: V \rightarrow W$ be a linear map, show that the kernel of T and range of T are subspaces of V and W respectively. 6
- (b) Consider $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ defined by

$$T(x, y, z) = (z, y, x, x + y + z)$$

 Write the matrix of T w.r.t. the standard ordered bases of \mathbb{R}^3 and \mathbb{R}^4 . 4
- (c) State and prove the rank-nullity theorem. 5
- (d) Find the rank and nullity of $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x, y, x + y)$.

Also, find the rank of the matrix of T w.r.t. the standard ordered basis of \mathbb{R}^3 . Verify whether rank of T is same as the rank of the matrix of T . $3+2=5$

UNIT—III

5. Answer any two questions :

2×2=4

(a) If $T:V \rightarrow W$ and $S:V \rightarrow W$ are two linear transformations, show that $T+S:V \rightarrow W$ is also a linear transformation.

(b) Show that a linear map $T:V \rightarrow W$ is one-one iff $T(v) = 0 \Leftrightarrow v = 0$.

(c) Justify if there can exist any one-one linear transformation from \mathbb{R}^4 to \mathbb{R}^3 .

6. Answer either [(a) and (b)] or [(c) and (d)] :

(a) Let V be a finite dimensional vector space. Let $T:V \rightarrow V$ be a linear transformation. Then show that the following are equivalent :

(i) $\ker(T) = \{0\}$

(ii) $\text{Im}(T) = V$, where $\text{Im}(T)$ is the image of T

(b) Show that every n -dimensional vector space over \mathbb{R} is isomorphic to \mathbb{R}^n .

(c) Let V and W be vector spaces over \mathbb{R} . Show that the set $L(V, W)$ of all linear transformations from V to W is a vector space with suitably defined addition and scalar multiplication.

(d) Let $T:V \rightarrow W$ be a linear map. Show that T is an isomorphism, if and only if, $T(B)$ is a basis of W whenever B is a basis of V .

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UNIT—IV

7. Answer any two questions :

2×2=4

(a) Find the eigenvalues of the linear operator $T:\mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (y, x)$.

(b) If u, v and w are the eigenvectors corresponding to the eigenvalues of

$$A = \begin{pmatrix} 1 & 8 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

justify whether u, v , and w are linearly independent.

(c) If v is an eigenvector of $T:V \rightarrow V$ corresponding to λ and $\alpha \in \mathbb{R}$ is a non-zero scalar, justify if αv is also an eigenvector of T corresponding to λ .

8. Answer either [(a) and (b)] or [(c) and (d)] :

(a) Show that the eigenvectors of a linear operator corresponding to distinct eigenvalues are linearly independent.

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- (b) Verify that the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}$$

obeys Cayley-Hamilton theorem. Hence find A^{-1} .

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- (c) Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

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- (d) Define invariant subspace under a linear operator. Give an example. Show that the intersection of two invariant subspaces under a linear operator T is also invariant under T . $1+1+3=5$

UNIT—V

9. Answer any two questions : $2 \times 2 = 4$

- (a) Define inner product space over the field of complex numbers.

- (b) Let $f: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f((x_1, y_1), (x_2, y_2)) = x_1 y_1 + x_2 y_2$$

Is f an inner product? Justify your answer.

- (c) Show that for any non-empty subset S of an inner product space V , the orthogonal complement S^\perp of S is a subspace of V .

10. Answer either [(a) and (b)] or [(c) and (d)] :

- (a) For any two vectors x, y in an inner product space, show that

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Give a geometrical interpretation of this identity.

4+1=5

- (b) Show that \mathbb{C}^2 is an inner product space over \mathbb{C} w.r.t. the inner product

$$\langle x, y \rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2$$

for $x = (x_1, x_2), y = (y_1, y_2)$ in \mathbb{C}^2 .

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- (c) Let V be an inner product space. Define $d: V \times V \rightarrow \mathbb{R}$ as

$$d(x, y) = \|x - y\| \quad \forall x, y \in V$$

Show that (V, d) is a metric space.

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- (d) Let W be a subspace of an inner product space V . Then prove that $V = W \oplus W^\perp$, i.e., V is an orthogonal direct sum of W and W^\perp .

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