2024/TDC (CBCS)/EVEN/SEM/
MTMHCC–601T/237

TDC (CBCS) Even Semester Exam., 2024

MATHEMATICS

(6th Semester)

Course No. : MTMHCC–601T

(Complex Analysis)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following : 2×2=4

(a) Prove that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$$

(b) Prove that $\operatorname{Re}(z) > 0$ and $|z-1| < |z+1|$ are equivalent.

(c) Find :

$$\operatorname*{Lt}_{z \to 2e^{\frac{\pi i}{2}}} \frac{z^3 + 8}{z^4 + 4z^2 + 16}$$

2. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Explain the geometrical interpretation of $\arg\left(\frac{z-\alpha}{z-\beta}\right)$. Hence find the condition for collinearity of three complex numbers z, α and β. 4+1=5

(b) Show that the equation of the straight line in the Argand plane can be put in the form $b\bar{z}+\bar{b}z=c$. 5

(c) (i) Find the loci of the point z satisfying the condition

$$\left|\frac{z-1}{z+1}\right|=2$$ 3

(ii) If the sum and product of two complex numbers are both real, then prove that the two numbers must be either real or conjugate. 2

(d) Define continuity. Prove that

$$f(z)=\frac{z}{z^4+1}$$

is continuous at all points in $|z|\le 1$ except at four points on the circle $|z|=1$. 1+4=5

UNIT—II

3. Answer any *two* of the following : 2×2=4

(a) Give an example to show that if a function is continuous at a point, then it is not differentiable.

(b) Define harmonic function.

(c) If u and v are conjugate harmonic functions, then prove that

$$dv=\frac{\partial u}{\partial x}dy-\frac{\partial u}{\partial y}dx$$

4. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Derive Cauchy-Riemann equations in polar form. 5

(b) Show that an analytic function in a domain with its derivative zero for every point the domain is constant. 5

(c) Find the analytic function $f(z)=u+iv$ of which $u=e^{-x}(x\sin y-y\cos y)$. 4

(d) (i) Show that the function $f(z)=\sqrt{|xy|}$, $z=x+iy$ is not regular at the origin although the Cauchy-Riemann equations are satisfied at the origin. 4

(ii) If $f(z)=u+iv$ and f' are both analytic functions, then show that both u and v are harmonic functions. 2

UNIT—III

5. Answer any *two* of the following : 2×2=4

(a) Show that

$$\left| \int_{\Gamma} \frac{dz}{z^2+1} \right| \leq \frac{\pi}{3}$$

where Γ denotes the circle $|z|=2$ in the 1st quadrant.

(b) State Cauchy's theorem.

(c) Evaluate

$$\oint_C (x^2 - iy^2)\,dz$$

along the parabola $y = 2x^2$ from (1, 1) to (2, 8).

6. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Prove Cauchy-Goursat theorem for a triangle. 6

(b) Use Cauchy's integral formula to evaluate

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$$

where C is the circle $|z|=3$. 4

(c) State and prove Cauchy's integral formula. 5

(d) (i) If $f(z)$ is analytic within a circle C, given by $|z-a|=r$ and if $|f(z)| \leq M$ on C, then prove that

$$|f^{(x)}(a)| \leq \frac{M \lfloor x}{R^n}$$ 3

(ii) Find the value of the integral

$$\int \frac{1}{z-a}\,dz$$

round a circle whose equation is $|z-a|=r$. 2

UNIT—IV

7. Answer any *two* of the following : 2×2=4

(a) State fundamental theorem of algebra.

(b) State Taylor's theorem.

(c) Show that the series $1+\frac{1}{3}+\frac{1}{9}+\cdots$ is convergent and find its sum.

8. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) State and prove Liouville's theorem. 5

(b) Expand the following function in a Taylor's series about $z=0$ and determine the region of convergence in each case : 2+3=5

(i) e^z

(ii) $\sin z$

(c) Show that the series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$$

is *(i)* convergent, if $p>1$ and *(ii)* divergent, if $p\le 1$. 5

(d) If the function $f(z)$ is analytic when $|z|<R$ and has the Taylor's expansion

$$\sum_{n=0}^{\infty} a_n z_n$$

then show that for $r<R$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^2$$ 5

UNIT—V

9. Answer any *two* of the following : 2×2=4

(a) Define removable singularity with an example.

(b) Define entire function with an example.

(c) Find the kind of singularities of the function $\sin z - \cos z$ at $z=\infty$.

10. Answer *either* [*(a)* and *(b)*] *or* [*(c)* and *(d)*] :

(a) State Cauchy's residue theorem. Evaluate the residues of

$$\frac{z^2}{(z-1)(z-2)(z-3)}$$

at $z=1, 2, 3$ and infinity, and show that their sum is zero. 2+4=6

(b) Expand

$$f(z)=\frac{1}{(z+1)(z+3)}$$

in a Laurent's series valid for region $1<|z|<3$. 4

(c) Use calculus of residues to prove that

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$ 5

(d) Prove that

$$\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\, d\theta = \frac{2\pi}{b^2}\{a-\sqrt{a^2-b^2}\}$$ 5

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