2023/TDC(CBCS)/EVEN/SEM/
MTMSEC–401T (A/B/C)/035

TDC (CBCS) Even Semester Exam., 2023

MATHEMATICS

(4th Semester)

Course No. : MTMSEC–401T

Full Marks : 50
Pass Marks : 20

Time : 3 hours

The figures in the margin indicate full marks for the questions

Candidates have to answer *either* from the Option—A *or* Option—B *or* Option—C

OPTION—A

Course No. : MTMSEC–401T (A)

(Graph Theory)

SECTION—A

Answer any *fifteen* of the following as directed : 1×15=15

1. Define a simple graph.

2. What is a complete graph?

3. Define pendant vertex.

4. The complete bipartite graph $K_{m,n}$ has _____ vertex and _____ edges.

(Fill in the blanks)

5. What is cut point of a graph?

6. What is a forest?

7. Draw the complete bipartite graph $K_{2,4}$.

8. What is a tree?

9. What is an Eulerian graph?

10. What is the degree of each vertex in an Eulerian graph?

11. Define a Hamiltonian graph.

12. Adjacency matrix of any graph is symmetric.

(Write True or false)

13. What is non-planar graph?

14. When a graph is said to be embedded in a surface S?

15. How many regions does a connected planar graph with n vertices and e edges have?

16. Give an example of planar connected graph such that $e = 3n - 6$.

17. What is the travelling salesman problem?

18. What is Dijkstra's algorithm used for?

19. Give an example of a weighted graph.

20. Floyd-Warshall algorithm is an example of dynamic programming approach.

(Write True or False)

SECTION—B

Answer any *five* of the following questions : 2×5=10

21. Suppose a simple graph has 15 vertices, 3 vertices of degree 4 and all other vertices of degree 3. Find the number of edges in the graph.

22. Define union of two graphs and give an example of it.

23. Prove that a cubic graph has a cut point if and only if it has a bridge.

24. A tree has $2n$ vertices of degree 1, $3n$ vertices of degree 2, and n vertices of degree 3. Determine the number of vertices and edges in the tree.

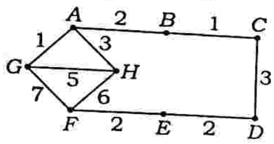
25. If G is a simple graph with $n (\geq 3)$ vertices and if $\deg v \geq \frac{n}{2}$ for each vertex v, then prove that G is Hamiltonian.

26. For which n, is K_n Eulerian?

27. Prove that K_5 is non-planar.

28. Verify Euler's formula for the complete bipartite graph $K_{2,5}$.

29. Solve the travelling salesman problem for the graph in the following figure :



30. Write the steps of Dijkstra's algorithm.

SECTION—C

Answer any *five* of the following questions : 5×5=25

31. Prove that the number of odd vertices in a graph is always even.

32. Show that the maximum number of edges in a complete bipartite graph of n vertices is $\frac{n^2}{4}$.

33. Prove that a graph G is a tree if and only if there is one and only one path between any two vertices of G.

34. Define adjacency matrix of a graph. Write four properties of adjacency matrix.

35. Let v be a point of a connected graph G. Prove that the following statements are equivalent :

(a) v is a cut point of G.

(b) There exist points u and w distinct from v such that v is on every $u-w$ path.

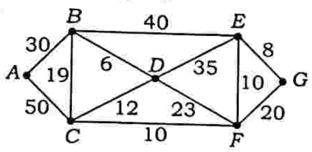
(c) There exists a partition of the set of point $V-\{v\}$ into subsets U and W such that for any points $u \in U$ and $w \in W$, the point v is on every $u-w$ path.

36. If G is a simple graph with $n (\geq 3)$ number of vertices and if $\deg v + \deg w \geq n$ for every pair of non-adjacent vertices v and w, then prove that G is Hamiltonian.

37. Prove that a connected planar graph with n vertices and e edges has $e - n + 2$ regions.

38. Let G be a maximal outer plan graph with $p \geq 3$ vertices all lying on the exterior face. Then, prove that G has $p - 2$ interior faces.

39. Use the shortest path algorithm to find a shortest path from A to G in the weighted graph of the following figure :



40. Illustrate Dijkstra's algorithm with an example.

(Continued)

OPTION—B

Course No. : MTMSEC–401T (B)

(Special Functions)

SECTION—A

Answer any *fifteen* of the following as directed : 1×15=15

1. All roots of $P_n(x) = 0$ are ____ and lie between ____ and ____.
(Fill in the blanks)

2. Write down the differential equation of the Legendre's equation.

3. Write down the general solution of the Legendre's equation.

4. When n is a positive integer, the value of $\frac{1}{\pi}\int_0^{\pi} [x \pm \sqrt{x^2 - 1}\cos\phi]^n \, d\phi$ is ____.
(Fill in the blank)

5. What is the complete solution of the Bessel's differential equation?

6. Write down the Bessel's differential equation of order 0.

(Turn Over)

7. What is the value of $\int_{-1}^{1} P_m(x) P_n(x) dx$, when $m \neq n$?

8. Write Bessel's function of the first kind of order $-n$.

9. What is the Laplace transform of e^{at}?

10. If $L\{F(t)\} = f(s)$, then $L\{e^{at}F(t)\} =$ _____.
(Fill in the blank)

11. Define inverse Laplace transform of $f(s)$.

12. What is inverse Laplace transform of $\frac{a}{s^2+a^2}$?

13. If $L(F(t)) = f(s)$, then what is $L\{F''(t)\}$, where (′) denotes differentiation w.r.t. t?

14. If $L(y(t)) = \bar{y}$, then obtain Laplace transform of $y''(t) + y(t)$, i.e., $L(y''(t) + y(t))$.

15. If $L(y(t)) = y(s)$, then obtain Laplace transform of $y''(t) + 2y'(t) + 5y(t)$, i.e., $L(y''(t) + 2y'(t) + 5y(t))$.

16. Write down the value of $L(t^n F(t))$, if $L(F(t)) = f(s)$.

17. What is the formula for infinite Fourier cosine transformation of $f(x)$?

18. Write down the finite Fourier sine transform of $f(x)$.

19. Write Fourier sine integral formula.

20. Write Fourier exponential integral formula.

SECTION—B

Answer any *five* of the following questions : 2×5=10

21. Show that $P_n(-1) = (-1)^n$.

22. Show that $P_n'(1) = \frac{n(n+1)}{2}$.

23. Prove that $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

24. Expand $J_0(x)$ and $J_1(x)$ in ascending powers of x, where $J_n(x)$ is a Bessel's function of first kind of order n.

25. Find Laplace transform of $t^5 e^{3t}$, i.e., $L(t^5 e^{3t})$.

26. Find inverse Laplace transform of

$$\left(\frac{1}{s-2}+\frac{2}{s+5}+\frac{6}{s^4}\right)$$

27. Using Laplace transformation, find $L\{\sinh at\}$.

28. Using Laplace transformation, find

$$L\{e^{-t}(3\sinh 2t-5\cosh 2t)\}$$

29. If $f_c(s)$ is the Fourier cosine transform of $F(x)$, then show that Fourier cosine transform of $F\left(\frac{x}{a}\right)$ is $af_c(as)$.

30. Define the relationship between Fourier transform and Laplace transform.

SECTION—C

Answer any *five* of the following questions : 5×5=25

31. Prove that

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx=\begin{cases}0, & m\neq n\\ \dfrac{2}{2n+1}, & m=n\end{cases}$$

where $P_n(x)$ is the Legendre's polynomial of first kind.

32. Prove that $nP_n(x)=xP_n'(x)-P_{n-1}'(x)$, where dashes denote differentials w.r.t. x.

33. Using Rodrigue's formula, express $f(x)=x^4+3x^3-x^2+5x-2$ in terms of Legendre polynomial.

34. Prove the following recurrence formulae for $J_n(x)$: 3+2=5

(a) $\dfrac{d}{dx}(x^n J_n(x))=x^n J_{n-1}(x)$

(b) $\dfrac{d}{dx}(x^{-n} J_n(x))=-x^{-n} J_{n+1}(x)$

35. (a) If $L\{F(t)\}=f(s)$, then prove that

$$L\{f(at)\}=\frac{1}{a}f\left(\frac{s}{a}\right)$$ 3

(b) Find $L\{\cos^2 at\}$, if $L\{F(t)\}=f(s)$. 2

36. (a) Show that

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}=\frac{1}{2a^3}\{\sin at-at\cos at\}$$ 3

(b) Find

$$L^{-1}\left\{\frac{s-2}{(s-2)^2+5^2}+\frac{s+4}{(s+4)^2+9^2}+\frac{1}{(s+2)^2+3^2}\right\}$$ 2

37. Apply Laplace transform to solve

$$\frac{d^2y}{dt^2}+y=6\cos 2t$$

Given that $y=3$, $\frac{dy}{dt}=1$, when $t=0$.

38. Using Laplace transform, solve the initial value problem

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=e^{-x}\sin x$$

where $y(0)=0$, $y'(0)=1$.

39. Solve the integral equation

$$\int_0^\infty F(x)\cos\lambda x\,dx=\begin{cases}1-\lambda, & 0\le\lambda\le 1\\ 0\ , & \lambda>1\end{cases}$$

40. Find $f(x)$ if Fourier cosine transform of $f(x)$ is $\frac{1}{1+s^2}$.

OPTION—C

Course No. : MTMSEC–401T (C)

(Vector Calculus/Vector Analysis)

SECTION—A

Answer any *fifteen* of the following as directed : $1\times15=15$

1. Write a vector equation of a straight line through the origin and parallel to $\vec{b}$.

2. Write the condition for four points to be coplanar.

3. When are two vectors $\vec{a}$ and $\vec{b}$ said to be perpendicular?

4. Write the vector equation of a plane passing through the origin and parallel to the vectors $\vec{a}$ and $\vec{b}$.

5. Express any vector function $f(t)$ in its component form.

6. Let $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vector functions of a scalar variable t. Then what is the value of $\frac{d}{dt}(\vec{u}\cdot\vec{v})$?

7. If $\vec{r} = (\cos nt)\hat{i} + (\sin nt)\hat{j}$, where n is a constant and t varies, then find $\frac{d^2\vec{r}}{dt^2}$.

8. Write the necessary and sufficient condition for a vector function $f(t)$ to have a constant direction.

9. Define divergence of a vector point function.

10. If $\vec{\nabla} = \hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}$, then find ∇^2.

11. If $\phi = 2x^3y^2z^4$, then find grad ϕ.

12. Define solenoidal vector.

13. If $\vec{a}(t) = t^2\hat{i} + 2t^3\hat{j} - 5\hat{k}$, then find $\int \vec{a}(t)\,dt$.

14. Write the value of $\int\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)dt$.

15. Write the value of $\int\left(\vec{a}\times\frac{d^2\vec{a}}{dt^2}\right)dt$.

16. What do you mean by line integrals?

17. Define potential energy.

18. What is the formula for tangential component of velocity?

19. Define principle of work.

20. A particle moves along the curve $x = e^t$, $y = 2\cos 2t$, $z = 2\sin 2t$, where t is the time. Find the velocity of the particle at time $t = 0$.

SECTION—B

Answer any *five* of the following questions : 2×5=10

21. Find the vector equation of a line joining the points whose position vectors are (1, −2, −1) and (0, −2, 3).

22. Prove that

$$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$$

23. If A and B are differential vector functions, then prove that

$$\frac{d}{dt}(A\cdot B) = A\cdot\frac{dB}{dt} + B\cdot\frac{dA}{dt}$$

24. If a vector function $f(t)$ be of constant magnitude, then prove that $f\cdot\frac{df}{dt} = 0$.

25. If $\phi = x^2 - y^2 + 4z$, then prove that $\nabla^2\phi = 0$.

26. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and c is any constant vector, then prove that $\text{div}\,\vec{r} = 3$ and $\text{curl}\,c = 0$.

27. If $f(t) = 3\hat{i} + (t^3 + 4t^7)\hat{j} + t\hat{k}$, then find $\int_1^2 f(t)\,dt$.

28. The equation of motion of a particle P of mass m is given by $m\dfrac{d^2\vec{r}}{dt^2} = f(r)r_1$, where $\vec{r}$ is the position vector of P measured from origin O, r_1 is the unit vector in the direction of $\vec{r}$ and $f(r)$ is a function of the distance of P from O. Show that $\vec{r} \times \dfrac{d\vec{r}}{dt} = C$, where C is a constant vector.

29. The acceleration of a particle at any time t is $e^t\hat{i} + e^{2t}\hat{j} + \hat{k}$. Find the velocity at time t, if the initial velocity be $\left(\hat{i} + \frac{1}{2}\hat{j}\right)$.

30. A particle moves along the curve $x = t$, $y = t^2$, $z = 2t^2$. Find the components of the velocity in the direction $(\hat{i} + \hat{j} + \hat{k})$ at time $t = 1$.

SECTION—C

Answer any *five* of the following questions : 5×5=25

31. *(a)* Find the vector equation of the plane passing through the points whose position vectors are (2, 3, −1), (3, 4, −2) and (1, −3, 2). 2

(b) Find the equation of a sphere whose centre is at a given point and whose radius is a. 3

32. *(a)* If $\vec{a}$, $\vec{b}$, $\vec{c}$ are the three non-coplanar vectors, then show that

$$[\vec{a}\times\vec{b} \;\; \vec{b}\times\vec{c} \;\; \vec{c}\times\vec{a}] = [\vec{a} \;\; \vec{b} \;\; \vec{c}]^2$$ 3

(b) If $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{c} = 3\hat{i} + 4\hat{j} - 2\hat{k}$, then find $(\vec{a}\times\vec{b})\times\vec{c}$. 2

33. Prove that a vector function $f(t)$ have constant direction if and only if

$$f \times \frac{df}{dt} = 0$$

34. *(a)* If $\vec{r} = (a\cos t)\hat{i} + (a\sin t)\hat{j} + (at\tan\alpha)\hat{k}$, then find $\left|\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right|$ and $\left[\dfrac{d\vec{r}}{dt} \;\; \dfrac{d^2\vec{r}}{dt^2} \;\; \dfrac{d^3\vec{r}}{dt^3}\right]$. 3

(b) If $A = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ and $B = (2t-3)\hat{i} + \hat{j} - t\hat{k}$, then find $\frac{d}{dt}\left(A \times \frac{dB}{dt}\right)$. 2

35. (a) If $\phi = 2x^3y^2z^4$, then prove that

$$\text{div} \cdot \text{grad} \cdot \phi = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$ 2

(b) Prove that

$$\text{curl} \cdot (u\vec{v}) = (\text{grad } u) \times \vec{v} + u\,\text{curl} \cdot \vec{v}$$ 3

36. (a) If f and g are vector functions, then prove that

$$\nabla(fg) = f\,\nabla g + g\,\nabla f$$ 3

(b) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then prove that

$$\text{grad} \cdot r^m = mr^{m-2}\vec{r}$$ 2

37. Let $A = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$. Evaluate $\int_C A \cdot dr$ from (0, 0, 0) to (1, 1, 1) along the path C given by—

(a) $x = t,\ y = t^2,\ z = t^3$;

(b) the straight line joining (0, 0, 0) and (1, 1, 1). 3+2

38. (a) Let $F = -3x^2\hat{i} + 5xy\hat{j}$. Evaluate $\int_C F \cdot dr$, where C is the curve in the xy-plane, $y = 2x^2$ from (0, 0) to (1, 2). 3

(b) Evaluate $\int_0^{\pi/2} (5\cos u\hat{i} - 7\sin u\hat{j})\,du$. 2

39. A particle moves according to the law $\vec{r} = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$. Find magnitude of the tangential and normal components of the velocity and acceleration.

40. What is the formula for work done in vector calculus? Find the work done by the force

$$F = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$$

in moving a particle in the xy-plane from (0, 0) to (1, 1) along the parabola $y^2 = x$. 1+4=5

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