2023/TDC(CBCS)/EVEN/SEM/
MTMHCC-602T/037

# TDC (CBCS) Even Semester Exam., 2023

## MATHEMATICS

( Honours )

( 6th Semester )

Course No. : MTMHCC-602T

( Linear Algebra )

*Full Marks* : 70

*Pass Marks* : 28

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

### SECTION—A

Answer any *ten* of the following questions : 2×10=20

**1.** Prove that in a vector space $V(F)$, $0 \cdot x = 0$, $\forall\, x \in V$.

**2.** Let $S = \{(1, 4), (0, 3)\}$ be a subset of $\mathbb{R}^2(\mathbb{R})$. Show that $(2, 3) \in L(S)$.

3. Prove that if $V(F)$ is a vector space of dimension $n$, then any $n+1$ vectors in $V$ are linearly dependent over $F$.

4. Examine whether the mapping $T:\mathbb{R}^3 \to \mathbb{R}$ defined by $T(x, y, z) = x^2 + y^2 + z^2$ is a linear transformation.

5. Find the nullity of the linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^3$ such that $T(x, y) = (x, x+y, y)$.

6. If $V$ is a finite dimensional vector space, prove that a linear transformation $T: V \to V$ is one-one if $T$ is onto.

7. Define isomorphism between two vector spaces and give an example.

8. Show that inverse of a linear transformation, when it exists, is again a linear transformation.

9. Prove that a linear transformation $T: V \to W$ is non-singular if $T$ carries each linearly independent subset of $V$ onto a linearly independent subset of $W$.

10. Define eigenvalue and eigenvector of a linear operator.





11. Define eigenspace of a linear operator $T: V \to V$ associated with an eigenvalue of it and prove that it is a subspace of $V$.

12. Define minimal polynomial of a linear operator.

13. Let $V$ be an inner product space. Show that $\langle u, v \rangle = 0$, for all $v \in V \Rightarrow u = 0$.

14. Using Cauchy-Schwarz inequality, prove that cosine of an angle is of absolute value at most 1.

15. Prove that an orthonormal set of non-zero vectors in an inner product space is linearly independent.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. *(a)* Prove that a necessary and sufficient condition for a non-empty subset $W$ of a vector space $V(F)$ to be a subspace is that $W$ is closed under vector addition and scalar multiplication. 5

(b) If $S_1$ and $S_2$ are two subsets of a vector space $V(F)$, prove that—

(i) $S_1 \subseteq S_2 \Rightarrow L(S_1) \subseteq L(S_2)$

(ii) $L(S_1 \cup S_2) = L(S_1) + L(S_2)$

(iii) $L(L(S_1)) = L(S_1)$ 1+2+2=5

**17.** (a) If $V$ is a finite dimensional vector space and $\{v_1, v_2, ..., v_r\}$ is a linearly independent subset of $V$, then prove that $\{v_1, v_2, ..., v_r\}$ can be extended to form a basis of $V$. 5

(b) Define dimension of a vector space. If $W$ is a subspace of a finite dimensional vector space $V(F)$, then prove that

$$\dim\left(\frac{V}{W}\right) = \dim V - \dim W$$ 1+4=5

**18.** (a) Define kernel and range of a linear transformation. If $T: V \to V$ is a linear operator, show that the following statements are equivalent : 1+1+3=5

(i) Range $(T) \cap$ Ker $(T) = \{0\}$

(ii) If $T(T(v)) = 0$, then $T(v) = 0$, $v \in V$





(b) Define rank and nullity of a linear transformation. Find the rank and nullity of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T(x, y, z) = (x+z, x+y+2z, 2x+y+3z)$$

1+1+3=5

**19.** (a) State and prove Sylvester's law of nullity. 5

(b) Define matrix of a linear transformation. Find the matrix of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (x+y, 2z-x)$$

with respect to the standard ordered basis of $\mathbb{R}^3$ and $\mathbb{R}^2$. 2+3=5

**20.** (a) Let $U(F)$ and $V(F)$ be two vector spaces and $T: V \to U$ be a linear transformation. Prove that

$$\frac{V}{\text{Ker } T} \cong \text{Range } T$$

5

(b) Let $V$ and $W$ be two vector spaces over a field $F$ of dimensions $m$ and $n$ respectively. Prove that Hom $(V, W)$ has dimension $mn$, where Hom $(V, W)$ is the vector space of all linear transformations from $V$ to $W$. 5

**21.** *(a)* If $A$ and $B$ are two subspaces of a vector space $V(F)$, then prove that
$$\frac{A+B}{A} \cong \frac{B}{A \cap B}$$ 5

*(b)* If $T_1$, $T_2 \in \text{Hom}(V, W)$, then show that

*(i)* $r(\alpha T_1) = r(T_1)$ for all $\alpha \in F$, $\alpha \neq 0$

*(ii)* $|r(T_1) - r(T_2)| \leq r(T_1 + T_2) \leq r(T_1) + r(T_2)$

where $r(T)$ means rank of $T$. 2+3=5

**22.** *(a)* Let $T$ be a linear operator on a finite dimensional vector space $V$ over a field $F$. Prove that $c \in F$ is an eigen-value of $T$ if and only if $T - cI$ is singular. 5

*(b)* State and prove Cayley-Hamilton theorem. 5

**23.** *(a)* Let $V$ be a finite dimensional vector space over the field $\mathbb{R}$ of real numbers and dim $V = 2$. Let $T$ be a linear operator on $V$ such that $T(v_1) = \alpha v_1 + \beta v_2$, $T(v_2) = \gamma v_1 + \delta v_2$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\{v_1, v_2\}$ is a basis of $V$. Find necessary and sufficient condition that 0 is an eigenvalue of $T$. 5





*(b)* Determine the eigenvalues and corresponding eigenvectors of the matrix
$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$ 5

**24.** *(a)* Let $V$ be an inner product space. Prove that $|\langle u, v\rangle| \leq \|u\|\|v\|$, for all $u, v \in V$. Also, prove that $|\langle u, v\rangle| \leq \|u\|\|v\|$ if and only if $u$ and $v$ are linearly dependent. 3+2=5

*(b)* Let $v$ be a non-zero inner product space of dimension $n$. Prove that $V$ has an orthonormal basis. 5

**25.** *(a)* State and prove Bessel's inequality. 5

*(b)* Let $W_1$ and $W_2$ be subspaces of a finite dimensional inner product space $V$. Show that

*(i)* $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$

*(ii)* $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$ 2½+2½=5

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