2023/TDC (CBCS)/EVEN/SEM/
MTMHCC–601T/036

TDC (CBCS) Even Semester Exam., 2023

MATHEMATICS

(Honours)

(6th Semester)

Course No. : MTMHCC–601T

(Complex Analysis)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following : 2×10=20

1. Prove that $|x|+|y|\leq\sqrt{2}|x+iy|$, where x and y are real.

2. If $|z_1|=|z_2|$ and $\arg z_1+\arg z_2=0$, then show that z_1 and z_2 are conjugate.

3. Find

$$\underset{z \to e^{\pi i/3}}{\text{Lt}} (z - e^{\pi i/3}) \frac{z}{z^3 + 1}$$

4. Write down the Cauchy-Riemann equations in Cartesian coordinates.

5. Define analytic function with an example.

6. Is the function $u(x, y) = 2xy + 3xy^2 - 2y^3$ harmonic?

7. Evaluate

$$\int_0^{1+i} (x^2 + iy)\, dz$$

along the path $y = x$.

8. Evaluate

$$\oint_C \frac{e^z}{z-2}\, dz$$

where C is the circle $|z| = 3$.

9. If $f(z)$ is analytic in a simply connected region D, then show that

$$\int_a^z f(z)\, dz$$

is independent of the paths in D joining two points a and z.

10. State Liouville's theorem.

11. Expand $f(z) = \sin z$ in a Taylor's series about $z = 0$.

12. Show that the series

$$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

is convergent.

13. Find the poles of

$$\left(\frac{z+1}{z^2+1}\right)^2$$

14. Define Laurent's theorem.

15. Define pole with an example.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. (a) If z lies on the circle $|z-1| = 1$, then prove that $\arg(z-1) = 2\arg z = \frac{2}{3}\arg z(z-1)$. 3

(b) Find the equation of the circle passing through the points z_1, z_2, z_3. 5

(c) Show that the zeros of $\sin z$ are real numbers. 2

17. *(a)* Show how the complex number $\frac{z_1}{z_2}$ can be represented by a vector. What is the vector representation of the conjugate of the complex number z? Deduce the condition for two complex numbers z_1 and z_2 to be conjugate. 5

(b) Prove that a function continuous in a closed and bounded region in the z plane is bounded. 5

18. *(a)* If a function is differentiable in the z plane, then prove that it is continuous in the z plane. Give an example to show that the converse is not true. 3+2=5

(b) If $f(z)$ is analytic single-valued complex function of $z = x + iy$, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$ 5

19. *(a)* Show that the function

$$f(z) = e^{-z^{-4}}, \quad z \neq 0$$
$$= 0, \quad z = 0$$

is not analytic at $z = 0$, although Cauchy- Riemann equations are satisfied at that point. 5

(b) Prove that $e^x \cos y$ is harmonic. Determine its harmonic conjugate and find the corresponding analytic function in terms of z. 5

20. *(a)* If $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points and on a simple closed curve C, then prove that

$$\oint_C f(z)\,dz = 0$$

Give an example to show that

$$\oint_C f(z)\,dz = 0$$

but $f(z)$ is not analytic. 4+3=7

(b) Evaluate

$$\oint_C |z|^2\,dz$$

where C is the square with vertices (0, 0), (1, 0), (1, 1), (0, 1). 3

21. *(a)* Prove that if $f(z)$ is integrable along a curve C having finite length L and if there exists a positive number M such that $|f(z)| \leq M$ on C, then

$$\left|\int_C f(z)\,dz\right| \leq ML$$ 4

(b) Determine the domain of analyticity of the function $f: \mathbb{C} \to \mathbb{C}$, $f(z) = ze^{-z}$ and apply Cauchy's theorem to show that

$$\int_{\Gamma} ze^{-z} dz = 0$$

where $\Gamma = \{z : |z| = 1\}$. 4

(c) State Cauchy's integral formula. 2

22. (a) State the fundamental theorem of algebra and prove it by using Liouville's theorem. 5

(b) Expand $\log(1+z)$ in a Taylor's series about $z = 0$ and determine the region of convergence for the resulting series. 5

23. (a) If $f(z)$ is continuous in a simply connected region D and

$$\oint_C f(z) dz = 0$$

where C is any simple closed curve in D, then show that $f(z)$ is analytic in D. 5

(b) Show that the positive term geometric series $1 + r + r^2 + \cdots$ converges for $r < 1$ and diverges to $+\infty$ for $r \geq 1$. 5

24. (a) Define different types of singularities with examples. 5

(b) State and prove Cauchy's residue theorem. 5

25. Use calculus of residues to prove that—

(a) $$\int_0^{2\pi} e^{\cos\theta} \cdot \cos(n\theta - \sin\theta) d\theta = \frac{2\pi}{\underline{|n}}$$

(b) $$\int_{-\pi}^{\pi} \frac{a\cos\theta}{a + \cos\theta} d\theta = 2\pi a\left[1 - \frac{a}{\sqrt{a^2 - 1}}\right], \quad a > 1$$

5+5=10

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