2022/TDC/ODD/SEM/
MTMHCC–502T/330

TDC (CBCS) Odd Semester Exam., 2022

MATHEMATICS

(Honours)

(5th Semester)

Course No. : MTMHCC–502T

(Multivariate Calculus)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following questions : 2×2=4

(a) Show that

$$\lim_{(x,\, y)\to(0,\, 0)} \frac{2xy^2}{x^2+y^4}$$

does not exist.

(b) Find the directional derivative of $f(x, y) = xe^y + \cos(xy)$ at the point (2, 0) in the direction of $u = 3i - 4j$.

(c) Find the gradient of the function $f(x, y, z) = \log_e(x^2 + y^2 + z^2)$ at the point (1, 2, 1).

Answer *either* Question No. **2** *or* **3** : 10

2. (a) Let

$$f(x, y) = \begin{cases} x\sin(\frac{1}{y}) + y\sin(\frac{1}{x}) & , \ xy \neq 0 \\ 0 & , \ xy = 0 \end{cases}$$

Show that the repeated limits of f do not exist whereas limit exists at (0, 0). 5

(b) Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & , \ (x, y) \neq (0, 0) \\ 0 & , \ (x, y) = (0, 0) \end{cases}$$

is continuous at the origin. 5

(Continued)

3. (a) If

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y} & , \ x \neq y \\ 0 & , \ x = y \end{cases}$$

show that f is discontinuous at the origin but possesses partial derivatives f_x and f_y at the origin. 5

(b) Let $f(x, y)$ be a function and $(a, b) \in \text{domain}(f)$ such that one of the partial derivatives f_x and f_y exists and is bounded in a neighbourhood of (a, b) and the other exists at (a, b). Prove that $f(x, y)$ is continuous at (a, b). 5

UNIT—II

4. Answer any *two* of the following questions : 2×2=4

(a) Show that $f(x, y) = y^2 + x^2y + x^4$ has a minimum at (0, 0).

(b) Show that the function $f(x, y) = 2x^4 - 3x^2y + y^2$ has neither a maximum nor a minimum at (0, 0).

(c) Give an example of a function $f(x, y)$ having an extreme value at (0, 0) even though the partial derivatives f_x and f_y do not exist at (0, 0).

(Turn Over)

Answer *either* Question No. **5** *or* **6** : 10

5. *(a)* Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the conditions $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$ and $z=x+y$. 5

(b) Prove that the volume of the greatest rectangular parallelopiped, that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is $\frac{8abc}{3\sqrt{3}}$. 5

6. *(a)* Show that the length of the axes of the section of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

by the plane $lx+my+nz=0$ are the roots of the quadratic in r^2

$$\frac{l^2a^2}{r^2-a^2}+\frac{m^2b^2}{r^2-b^2}+\frac{n^2c^2}{r^2-c^2}=0$$ 5

(b) Show that if $2x+3y+4z=a$, the maximum value of $x^2y^3z^4$ is $\left(\frac{a}{9}\right)^9$. 5

UNIT—III

7. Answer any *two* of the following questions : 2×2=4

(a) Find the curl of

$$\vec{F}=(x^2-z)i+xe^z j+xyk$$

(b) Define divergence of a vector field at a point. Find the divergence of $\vec{F}(x,y)=\frac{-y}{x^2+y^2}i+\frac{x}{x^2+y^2}j$ at all the points in the domain.

(c) Evaluate $\iint(x^2+y)\,dx\,dy$ over the rectangle $[0, 1;\ 0, 2]$.

Answer *either* Question No. **8** *or* **9** : 10

8. *(a)* Evaluate $\iint_R (y-2x)\,dx\,dy$ over $R=[1, 2;\ 3, 5]$. 5

(b) Prove that

$$\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy = \frac{1}{2} \text{ and}$$
$$\int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx = -\frac{1}{2}$$ 5

9. (a) Find $\iint y\,dx\,dy$ over the part of the plane bounded by the line $y = x$ and the parabola $y = 4x - x^2$. 5

(b) By changing to polar coordinates, show that $\iint_E \sqrt{x^2 + y^2}\, dx\, dy = \frac{38\pi}{3}$, where E is the region in the xy-plane bounded by $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. 5

UNIT—IV

10. Answer any *two* of the following questions : 2×2=4

(a) Evaluate $\int_0^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_0^a dz$ by passing over to cylindrical coordinates.

(b) Change the order of the integration

$$\int_0^1 dx \int_x^{\sqrt{x}} f(x, y)\, dy$$

(c) Evaluate the line integral $\int_C xy\,dx$, where C is the arc of the parabola $x = y^2$ from (1, – 1) to (1, 1).

Answer *either* Question No. **11** *or* **12** : 10

11. (a) Compute $\iiint_E xyz\,dx\,dy\,dz$, where E is bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. 5

(b) Evaluate $\int_C (x-y)^3 dx + (x-y)^3 dy$, where C is the circle $x^2 + y^2 = a^2$ in the counter-clockwise direction. 5

12. (a) Find the volume cut from a sphere of radius a by a right circular cylinder with b as radius of the base and whose axis passes through the centre of the sphere. 5

(b) Show that

$$\int_C (y^2 + z^2) dx + (z^2 + x^2) dy + (x^2 + y^2) dz = -2\pi ab^2$$

where the curve C is the part for which $z \geq 0$ of the intersection of the surfaces $x^2+y^2+z^2=2ax$, $x^2+y^2=2bx$, $a>b>0$, the curve begins at the origin and runs at first in the positive octant. 5

UNIT—V

13. Answer any *two* of the following questions : 2×2=4

(a) Evaluate the line integral $\int_C (x^2 dx + xy\, dy)$ taken along the line segment from (1, 0) to (0, 1).

(b) State Stokes' theorem.

(c) State Green's theorem in $\mathbb{R}^2$.

Answer *either* Question No. **14** *or* **15** : 10

14. (a) Verify Green's theorem in the plane for $\oint_C (xy+y^2)dx + x^2 dy$, where C is the closed curve of the region bounded by $y=x$ and $y=x^2$. 5

(b) Use Stokes' theorem to find the line integral $\int_C x^2y^3dx + dy + zdz$, where C is the circle $x^2+y^2=a^2$, $z=0$. 5

15. (a) Evaluate the surface integral

$$\iint_S x\,dy\,dz + dz\,dx + xz^2 dx\,dy$$

where S is the outer part of the sphere $x^2+y^2+z^2=1$ in the first octant. 5

(b) Use Gauss divergence theorem to evaluate

$$\iint_S y^2 z\,dx\,dy + xz\,dy\,dz + x^2 y\,dz\,dx$$

where S is the outer side of the surface in the first octant formed by the paraboloid of revolution $z=x^2+y^2$, cylinder $x^2+y^2=1$ and the coordinate planes. 5

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