2023/TDC(CBCS)/ODD/SEM/
MTMHCC–502T/311

TDC (CBCS) Odd Semester Exam., 2023

MATHEMATICS

(Honours)

(5th Semester)

Course No. : MTMHCC–502T

(Multivariate Calculus)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer *ten* questions, selecting *two* from each Unit : 2×10=20

UNIT—I

1. Show that

$$\lim_{(x,\, y)\to(0,\, 0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}=0$$

2. If $f(x, y) = \sqrt{|xy|}$, find $f_x(0, 0)$ and $f_y(0, 0)$.

3. Find the tangent plane to the surface $z = x\cos y - ye^x$ at $(0, 0, 0)$.

UNIT—II

4. Find the local extreme values of

$$f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$$

5. Find the critical points of the function

$$f(x, y) = 10xye^{-(x^2+y^2)}$$

and identify saddle point, if any.

6. Give an example with justification of a function $f(x, y)$ which has an extreme value at a point even though the partial derivatives f_x and f_y do not exist thereat.

UNIT—III

7. Define divergence of a vector field. Find the divergence of $\vec{F}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$.

8. Find the curl of $\vec{F} = (x^2 - z)\hat{i} + xe^z\hat{j} + xy\hat{k}$.

9. Evaluate $\iint y\,dxdy$ over the part of the plane bounded by the lines $y = x$ and the parabola $x^2 + y = 4x$.

UNIT—IV

10. Find the value of

$$\int_C \{(x + y^2)\,dx + (x^2 - y)\,dy\}$$

taken in the clockwise sense along the closed curve C formed by $y^3 = x^2$ and the chord joining $(0, 0)$ and $(1, 1)$.

11. Compute the integral

$$\iiint_E xyz\;dxdydz$$

over a domain bounded by $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

12. Evaluate the integral

$$\int_0^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_0^a dz$$

by passing over to cylindrical coordinates.

UNIT—V

13. Define surface integral of the second type.

14. Using the line integral, compute the area of the loop of Descarte's folium $x^3 + y^3 = 3axy$.

15. State Gauss' divergence theorem.

SECTION—B

Answer *five* questions, selecting *one* from each Unit : 10×5=50

UNIT—I

16. *(a)* Show that the simultaneous limit and both the repeated limits exist when $(x, y) \to (0, 0)$ for the function

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 \quad , & (x, y) = (0, 0) \end{cases}$$ 5

(b) Show that the function f is continuous at the origin, where

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 \quad , & (x, y) = (0, 0) \end{cases}$$ 5

17. *(a)* Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & x^2 + y^2 \neq 0 \\ 0 \quad , & x = 0 = y \end{cases}$$

possesses first partial derivatives everywhere, including the origin, but the function is discontinuous at the origin. 5

(b) State and prove a sufficient condition for a function $f(x, y)$ of two variables to be continuous at a point (a, b). 5

UNIT—II

18. *(a)* Find the shortest distance from the origin to the hyperbola

$$x^2 + 8xy + 7y^2 = 225, \; z = 0$$ 5

(b) Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the conditions

$$\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$$

and $z = x + y$. 5

19. *(a)* Prove that the volume of the greatest rectangular parallelopiped, that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $\dfrac{8abc}{3\sqrt{3}}$. 5

(b) If $xyz = a^2(x + y + z)$, then show that the minimum value of $xy + yz + zx$ is $9a^2$. 5

UNIT—III

20. *(a)* Evaluate $\iint (y-2x)\,dxdy$ over the rectangle $R = [1, 2; 3, 5]$. 5

(b) Evaluate

$$\iint_E x^{m-1} y^{n-1} (1-x-y)^{p-1}\,dxdy,\ m \geq 1,\ n \geq 1,\ p \geq 1$$

where E is the region bounded by $x = 0$, $y = 0$, $x + y = 1$. 5

21. *(a)* Evaluate

$$\iint \sqrt{\frac{a^2b^2 - b^2x^2 - a^2y^2}{a^2b^2 + b^2x^2 + a^2y^2}}\,dxdy$$

over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. 5

(b) Evaluate : 5

$$\int_0^{\pi}\int_0^{\pi} |\cos(x+y)|\,dxdy$$

UNIT—IV

22. *(a)* Compute the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$ 5

(b) Evaluate

$$\iiint_E (y^2z^2 + z^2x^2 + x^2y^2)\,dxdydz$$

taken over the domain bounded by the cylinder $x^2 + y^2 = 2ax$ and the cone $z^2 = k^2(x^2 + y^2)$. 5

23. *(a)* Show that

$$\int_C (y^2 + z^2)\,dx + (z^2 + x^2)\,dy + (x^2 + y^2)\,dz = -2\pi ab^2$$ 5

(b) Compute

$$\iiint \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\,dxdydz$$

taken over the region $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$. 5

UNIT—V

24. *(a)* State and prove Green's theorem in the plane R^2. 5

(b) Evaluate

$$\iint_S (x\cos\alpha + y\cos\beta + z^2\cos\gamma)\,dS$$

where S denotes the closed surface bounded by the cone $x^2 + y^2 = z^2$ and the plane $z = 1$; and $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are direction cosines of the outward drawn normal of S. 5

25. *(a)* Show that

$$\iint_S (y-z)\,dydz + (z-x)\,dzdx + (x-y)\,dxdy = a^3\pi$$

where S is the portion of the surface $x^2 + y^2 - 2ax + az = 0,\ z \geq 0.$ 5

(b) Evaluate

$$\iint_S (x\,dydz + dzdx + xz^2\,dxdy)$$

where S is the outer side of the part of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant. 5

★★★