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2022/TDC/ODD/SEM/
MTMHCC–501T/329

TDC (CBCS) Odd Semester Exam., 2022

MATHEMATICS

(Honours)

(5th Semester)

Course No. : MTMHCC–501T

(Topology)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following questions : 2×2=4

(a) Let (X, d) be a metric space and x, y, z be any three points of X. Then prove that $d(x, y) \geq |d(x, z) - d(z, y)|$.

(b) Let (X, d) be a discrete metric space. Describe open sets for d.

(c) Prove that in a metric space (X, d), for $A, B \subset X$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

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2. Answer any *one* of the following questions : 10

(a) *(i)* Let (X, d) be a metric space and let $d^* : X \times X \to \mathbb{R}$ be defined by

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \ \forall\, x, y \in X$$

Prove that (X, d^*) is a metric space. 5

(ii) Let (X, d) be a metric space and $A \subset X$. Prove that int A, the interior of A, is the largest open set contained in A. 5

(b) *(i)* Let (X, d) be a metric space and let $\{G_i : 1 \le i \le n, n < \infty\}$ be a finite collection of closed sets in X. Prove that $\bigcup_{i=1}^{n} G_i$ is also a closed set in X. Give an example to show that union of an infinite collection of closed sets in a metric space is not necessarily closed. 3+2=5

(ii) Prove that a subset A of a metric space X is closed iff it contains all its limit points. 5

UNIT—II

3. Answer any *two* of the following questions : 2×2=4

(a) Define a Cauchy sequence in a metric space (X, d) and give an example.

(b) Define a complete metric space. Give an example of a metric space which is not complete.

(c) Prove that if a Cauchy sequence in a metric space has a convergent subsequence, then the sequence is convergent.

4. Answer any *one* of the following questions : 10

(a) *(i)* Let (X, d) be a metric space and let Y be a subspace of X. Prove that Y with the induced metric $d|Y$ is complete iff Y is closed in (X, d). 5

(ii) Let (X, d) and (Y, ρ) be metric spaces. Prove that a function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if for every sequence $\langle x_n \rangle$ in X converging to x_0, the sequence $\langle f(x_n) \rangle$ in Y converges to $f(x_0)$. 5

(b) *(i)* Let $C[0, 1]$ be the set of all the real-valued continuous functions defined on $[0, 1]$. Prove that $(C[0, 1], d)$ is a complete metric space, where d is defined as

$$d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|, \ \forall\, f, g \in C[0, 1]$$

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(ii) Let X and Y be metric spaces and let A be a non-empty subset of X. If f and g are continuous functions of X into Y such that $f(x) = g(x)$, $\forall\, x \in A$, then prove that $f(x) = g(x)$, $\forall\, x \in \overline{A}$. 5

UNIT—III

5. Answer any *two* of the following questions : 2×2=4

(a) Let (X, τ) be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. Find all the closed sets in X.

(b) Let X be any non-empty set and let $\tau = \{\phi, A, B, X\}$, where A and B are non-empty distinct proper subsets of X. Find for what conditions A and B must satisfy in order that τ may be a topology for X.

(c) In the topological space $(\mathbb{R}, u)$, where u is the usual topology on $\mathbb{R}$, prove that every open interval is an open set.

6. Answer any *one* of the following questions : 10

(a) (i) Let τ be the collection of subsets of $\mathbb{N}$ consisting of empty set ϕ and all subsets of the form $G_m = \{m, m+1, m+2, \cdots\}$, $m \in \mathbb{N}$. Show that τ is a topology on $\mathbb{N}$. What are the open sets containing 5? 4+1=5

(ii) Define cofinite topology on an infinite set X and prove that it is in fact a topology on X. 1+4=5

(b) (i) Let $\{\tau_\lambda : \lambda \in \Lambda\}$, where Λ is an index set, be a collection of topologies on X. Prove that $\bigcap\{\tau_\lambda : \lambda \in \Lambda\}$ is also a topology on X. Is the union of two topologies on a set is always a topology on X? Justify. 3+2=5

(ii) Define discrete and indiscrete topological spaces. Give an example of a topological space other than the discrete and indiscrete spaces in which open sets are exactly same as the closed sets with complete justification. 2+3=5

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UNIT—IV

7. Answer any *two* of the following questions : 2×2=4

(a) Define a Hausdorff space and give an example.

(b) Let A and B be two subsets of a topological space X. If $A \subseteq B$, then prove that $D(A) \subseteq D(B)$, where $D(A)$ represents the derived set of A.

(c) Find the interior, exterior and boundary of the set (0, 1) in $(\mathbb{R}, u)$, where u is the usual topology on $\mathbb{R}$.

8. Answer any *one* of the following questions : 10

(a) (i) Prove that every metric d on a non-empty set X induces a topology τ_d on X. Give an example of a topological space with justification which is not metrizable. 3+2=5

(ii) Show that every discrete topological space is Hausdorff. Give an example of a Hausdorff space which is not discrete. 4+1=5

(b) (i) If A is a subset of a topological space, then prove that $\overline{A} = A \cup D(A)$. 5

(ii) Let (X, τ) be a topological space and let A be a subset of X. Prove that $\overline{A}$ is the smallest closed set containing A. 5

UNIT—V

9. Answer any *two* of the following questions : 2×2=4

(a) Define a convergent sequence in a topological space. Give an example of a sequence in a topological space which has multiple limits.

(b) Define a continuous function from a topological space (X, τ_1) to another topological space (Y, τ_2).

(c) Prove that a function from a discrete space to any topological space is continuous.

10. Answer any *one* of the following questions : 10

(a) (i) Let τ denote the discrete topology on a non-empty set X. Show that a sequence (x_n) in (X, τ) is convergent iff there exists $x_0 \in X$ and there exists $N \in \mathbb{N}$ such that $x_n = x_0 \; \forall \; n \geq N$. 5

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(ii) Prove that a function f from a topological space X into another topological space Y is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$, for every $A \subset X$. 5

(b) (i) Let τ_f and u denote the cofinite topology and the usual topology respectively on $\mathbb{R}$. Show that the function $i : (\mathbb{R}, u) \to (\mathbb{R}, \tau)$ given by $i(x) = x \ \forall \ x \in \mathbb{R}$, is continuous. 5

(ii) Prove that in a Hausdorff space, a convergent sequence has unique limit. 5

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