2020/TDC(CBCS)/ODD/SEM/MTMHCC–501T/331

TDC (CBCS) Odd Semester Exam., 2020 held in March, 2021

MATHEMATICS

(5th Semester)

Course No. : MTMHCC–501T

(Topology)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

1. Answer any *ten* of the following questions : 2×10=20

(a) Define a metric space with example.

(b) Show that every open set in a metric space is a union of open spheres.

(c) Show that a set A in a metric space is closed iff $A = \overline{A}$.

(d) Give an example of a non-empty collection of non-empty open sets whose intersection is closed.

(e) Prove that any convergent sequence in a discrete metric space has only finitely many distinct terms.

(f) Justify whether $(0, 1)$ with the usual metric $d(x, y) = |x - y|$ is complete.

(g) Check if the function $f : (\mathbb{R}, u) \to (\mathbb{R}, d)$ defined by $f(x) = x \ \forall x \in \mathbb{R}$ is continuous. Here, u is the usual metric on $\mathbb{R}$ and d is the discrete metric on $\mathbb{R}$.

(h) Define convergence of a sequence in a metric space.

(i) Define co-finite topology on a non-empty set X. What happens if X is finite?

(j) Give example to justify that arbitrary union of closed sets in a topological space need not be closed.

(k) Define two topologies on the set $X = \{1, 2, 3, 4\}$ such that one is weaker than the other.

(l) Define relative topology.

(Continued)

(m) Justify whether the union of two topologies on a set is a topology.

(n) If A and B are sets in a topological space X, show that $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$.

(o) Show that in any topological space

$$A \text{ is open} \Leftrightarrow A = \text{int}(A)$$

(p) Define limit point and boundary point of a set in a topological space.

(q) Define convergence of a sequence in a topological space. Give example to show the non-uniqueness of limit.

(r) Define continuity of a function in topological space. Give an example.

(s) Show that any function from a discrete topological space to any other topological space is continuous.

(t) Let c be the co-finite topology on $\mathbb{R}$ and u be the usual topology on $\mathbb{R}$. Check if the function $f : (\mathbb{R}, c) \to (\mathbb{R}, u)$ defined by $f(x) = x \ \forall x \in \mathbb{R}$ is continuous.

(Turn Over)

SECTION—B

Answer *any* **five** questions

2. *(a)* Let (X, d) be a metric space. Let $\bar{d}$ be defined by

$$\bar{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \quad \forall x, y \in X$$

Show that $(X, \bar{d})$ is a metric space. 5

(b) Show that arbitrary union of open sets in a metric space is open. Justify if the same is true for intersection. 4+1=5

3. *(a)* Show that any finite set in a metric space is closed. 4

(b) Let A and B be subsets of a metric space X. Prove that—

(i) $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$

(ii) $\text{int}(A) \cap \text{int}(B) = \text{int}(A \cap B)$

Give example to show that

$$\text{int}(A) \cup \text{int}(B) \neq \text{int}(A \cup B)$$

in general. 1+2+1=4

(c) Describe all the open spheres in a discrete metric space. 2

4. *(a)* Prove the uniqueness of the limit of a convergent sequence in a metric space. 4

(b) Let X be a complete metric space and let Y be a subspace of X. Show that Y is complete iff Y is closed. 6

5. *(a)* Let X and Y be metric spaces and $f : X \to Y$. Then prove that f is continuous at $x_0 \in X$ if and only if for every sequence $< x_n >$ in X converging to x_0, the sequence $< f(x_n) >$ converges to $f(x_0)$. 6

(b) Show that every convergent sequence in a metric space is Cauchy. 4

6. *(a)* Let X be a non-empty set and $x \in X$ be a point. Let T be the collection consisting of ϕ and all those subsets of X that contain x. Show that T is a topology on X. 5

(b) Define co-countable topology on a set and show that it is a topology. 5

7. *(a)* Describe all the closed sets in a co-finite topology and in a co-countable topology. 3

(b) Let T be the collection consisting of $\mathbb{N}$ and all its finite subsets. Justify if T is a topology on $\mathbb{N}$. 2

(c) Define lower limit topology on $\mathbb{R}$. Establish that it is a topology. 5

8. (a) If T_1 and T_2 be two topologies on a non-empty set X, then show that $T_1 \cap T_2$ is also a topology on X. 5

(b) Show that every metric space is a Hausdorff space. 5

9. (a) Let A and B be any two sets in a topological space X. Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Give an example to show that in general $\overline{A} \cap \overline{B} \neq \overline{A \cap B}$. 3+1+1=5

(b) Show that the following are equivalent : 5

(i) int(A) is the union of all open subsets of A

(ii) int(A) is the largest open subset of A

10. (a) Let $f: X \to Y$ be a map from one topological space into another. Show that f is continuous iff $f^{-1}(F)$ is closed in X whenever F is closed in Y. 5

(b) If X is a Hausdorff space, then show that the limit of a convergent sequence is unique. 5

11. (a) Let f be a bijection from a topological space to another. Show that f is homeomorphism iff both f and f^{-1} are continuous. 5

(b) Let X, Y, Z be topological spaces. If $f: X \to Y$ and $g: Y \to Z$ are continuous, show that $g \circ f: X \to Z$ is continuous. 5

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