TDC (CBCS) Odd Semester Exam., 2023

MATHEMATICS

(Honours)

(5th Semester)

Course No. : MTMHCC-501T

(Topology)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

All symbols and notations used are as per the notations given in the book, 'Introduction to Topology and Modern Analysis' by G. F. Simmons

SECTION—A

Answer *ten* questions, selecting *two* from each Unit : 2×10=20

UNIT—I

1. Let X be a metric space. If $\{x\}$ is a subset of X consisting of a single point, then show that its complement $\{x\}'$ is open in X.

2. Consider $\mathbb{R}$ with the discrete metric d. Find out all dense subsets of $\mathbb{R}$.

3. Prove or disprove :

In any metric space $S_r[x] = \overline{S_r(x)}$.

UNIT—II

4. Let $\{x_n\}$ be a sequence in a metric space (X, d) such that $\{x_n\}$ converges to some $x_0 \in X$. Show that

$$\{n \in \mathbb{N} : d(x_n, x) \geq 10\}$$

is finite.

5. Consider the usual metric space $(\mathbb{R}, d)$. Show that the sequence $\left\{\frac{1}{n}\right\}$ cannot converge to 1.

6. Prove or disprove :

If x_0 is the limit of a convergent sequence $\{x_n\}$ in a metric space (X, d), then x_0 is a limit point of the set $\{x_n : n \in \mathbb{N}\}$.

UNIT—III

7. Write down all the topologies on $\{1, 2\}$.

8. Show that the set of all open intervals in $\mathbb{R}$ is not a topology on $\mathbb{R}$.

9. Is $\{G \subseteq \mathbb{R} : (0, 1) \not\subseteq G\} \cup \{\mathbb{R}\}$ a topology on $\mathbb{R}$? Justify.

(Continued)

UNIT—IV

10. Show that the indiscrete topology on $\mathbb{R}$ is not metrizable.

11. Consider $\mathbb{R}$ with the cofinite topology τ. Show that 5 is a limit point of $\mathbb{N}$ in $(\mathbb{R}, \tau)$.

12. Define a perfect set in a topological space. Give an example of a perfect set.

UNIT—V

13. Define convergence of a sequence in a topological space (X, τ).

14. Consider $\mathbb{R}$ with the cofinite topology τ. Does the sequence $\left\{\frac{1}{n}\right\}$ converge to 100 in $(\mathbb{R}, \tau)$? Justify.

15. Let τ_1 and τ_2 denote the discrete and usual topologies on $\mathbb{R}$ respectively. Show that every function $f : (\mathbb{R}, \tau_1) \rightarrow (\mathbb{R}, \tau_2)$ is continuous.

(Turn Over)

SECTION—B

Answer *five* questions, selecting *one* from each Unit : 10×5=50

UNIT—I

16. *(a)* Let (X, d) be a metric space. Show that $d_1 : X \times X \to \mathbb{R}$ defined by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)} \ \forall x, y \in X$$

is a metric on X. 5

(b) Let A be a subset of a metric space (X, d). Show that

$$\overline{A} = \{x \in X : d(x, A) = 0\}$$ 5

17. *(a)* Let (X, d) be a metric space. Show that a subset G of X is open if and only if G is a union of open spheres. 5

(b) Show that in a metric space, every closed sphere is a closed set. Is the converse true? Justify. 4+1=5

UNIT—II

18. *(a)* Let X be a complete metric space and let Y be a subspace of X. Show that if Y is closed in X, then Y is complete. Does the conclusion follow if the completeness of X is dropped? Justify. 3+2=5

(b) Let X and Y be metric spaces and f be a mapping of X into Y. If f is a constant mapping, then show that f is continuous. Use this to show that a continuous mapping need not have the property that image of every open set is open. 2+3=5

19. *(a)* Show that a Cauchy sequence in a metric space is convergent iff it has a convergent subsequence. 5

(b) Let X and Y be metric spaces and A be a non-empty subset of X. If f and g are continuous mappings of X into Y such that $f(x) = g(x) \ \forall x \in A$, then show that

$$f(x) = g(x) \ \forall x \in \overline{A}$$ 5

UNIT—III

20. *(a)* Show that

$$\tau = \{G \subseteq \mathbb{R} : 0 \in G\} \cup \{\phi\}$$

is a topology on $\mathbb{R}$. 4

(b) Prove or disprove : 3

$\{G \subseteq \mathbb{R} : G$ is finite or $\mathbb{R} \smallsetminus G$ is finite$\}$ is a topology on $\mathbb{R}$ ($\mathbb{R} \smallsetminus G$ stands for complement of G in $\mathbb{R}$).

(c) Can we compare the usual topology and the cofinite topology on $\mathbb{R}$? Justify. 3

21. (a) Let (X, τ) be a topological space and let $\{F_\alpha : \alpha \in \Lambda\}$ be a family of closed subsets of X. Is $\bigcup_{\alpha \in \Lambda} F_\alpha$ closed? Justify. 4

(b) Consider a non-empty set X. Show that the cofinite topology on X is same as the discrete topology on X if and only if X is finite. 6

UNIT—IV

22. (a) Let X be a topological space and A be an arbitrary subset of X. Show that

$$\overline{A} = \{x \in X : \text{each neighbourhood of } x \text{ intersects } X\}$$ 5

(b) Let X be an infinite set and τ be the cofinite topology on X. Show that (X, τ) is not metrizable. 5

23. (a) Let X be a topological space and A be an arbitrary subset of X. Show that A is closed if and only if A contains its boundary. 5

(b) Show that every metrizable topological space is Hausdorff. 5

UNIT—V

24. (a) Let (X, τ_1) and (Y, τ_2) be topological spaces. Show that $f : (X, \tau_1) \to (Y, \tau_2)$ is continuous iff

$$f(\overline{A}) \subset \overline{f(A)} \quad \forall A \subseteq X$$ 5

(b) Prove or disprove :

In every topological space, every convergent sequence has a unique limit. 5

25. (a) Consider $\mathbb{R}$ with the co-countable topology τ. Does the sequence $\left\{\frac{1}{n}\right\}$ converge in $(\mathbb{R}, \tau)$? Justify. 5

(b) Let τ_1 and τ_2 be topologies on a non-empty set X such that the identity map $i : (X, \tau_1) \to (X, \tau_2)$ is continuous. Show that $\tau_2 \subseteq \tau_1$. 3

(c) Give examples of two topologies τ_1 and τ_2 on a non-empty set X such that the identity map $i : (X, \tau_1) \to (X, \tau_2)$ is not continuous. 2

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