TDC (CBCS) Even Semester Exam., 2023

MATHEMATICS

(Honours)

(4th Semester)

Course No. : MTMHCC–402T

(Riemann Integration and Series of Functions)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following questions : 2×10=20

1. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 \ , \text{ when } x \in \mathbb{Q} \cap [0, 1] \\ 1 \ , \text{ when } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1] \end{cases}$$

For any partition P of $[0, 1]$, find $L(P, f)$.

2. Give example of a Riemann integrable function $f:[0,1]\to\mathbb{R}$ which is not monotone.

3. Show that $\int_0^1 5\,dx$ exists and find its value.

4. Construct a function $f:\mathbb{R}\to\mathbb{R}$ such that $|f|$ is Riemann integrable but f is not Riemann integrable.

5. Is the equation $\int_a^b f'(x)\,dx = f(b)-f(a)$ always valid for any differentiable function $f:[a,b]\to\mathbb{R}$? Justify.

6. State fundamental theorem of integral calculus.

7. Test the convergence of the integral

$$\int_0^1 \frac{dx}{x^2}$$

8. Write down Legendre's duplication formula about gamma function.

9. Define beta function and gamma function. 1+1=2

10. Find the pointwise limit of the sequence of functions $\{\tan^{-1} nx\}$, $n\geq 1$, $x\geq 0$.

11. State Weierstrass M-test for convergence of a series of functions.

12. Let $\langle f_n\rangle$ be a sequence of functions defined on $[a,b]$ and $f:[a,b]\to\mathbb{R}$. Explain the meaning of $f_n\to f$ pointwise and $f_n\to f$ uniformly as $n\to\infty$. 1+1=2

13. Define limit superior of a sequence $\{a_n\}$.

14. Find the radius of convergence of the power series $1+x+2!x^2+3!x^3+\cdots$.

15. Does there exist a power series $\sum_{n=0}^{\infty} a_n x^n$ which converges for $x=3$ but diverges for $x=-2$? Justify your answer.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. (a) Let $f:[a,b]\to\mathbb{R}$ be bounded and P_1, P_2 are any two partitions of $[a,b]$. Show that $L(P_1, f)\leq U(P_2, f)$. 5

(b) Show that every continuous function $f:[a,b]\to\mathbb{R}$ is Riemann integrable. Give an example to show that the converse is not true. 3+2=5

17. (a) Show that a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable over $[a, b]$ iff $\forall\ \varepsilon > 0$ there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$. 5

(b) Show that

$$f(x) = \begin{cases} \frac{1}{2^n}, & \text{when } \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n} \\ 0, & \text{when } x = 0 \end{cases}$$

is Riemann integrable on $[0, 1]$ and hence find $\int_0^1 f(x)dx$. 2+3=5

18. (a) If $f : [a, b] \to \mathbb{R}$ is bounded and Riemann integrable on $[a, b]$, then show that $|f|$ is also Riemann integrable on $[a, b]$ and that

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f|(x)\,dx$$ 3+2=5

(b) Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$ 5

19. (a) If $f, g : [a, b] \to \mathbb{R}$ are integrable, then show that $f - g$ is also integrable and

$$\int_a^b (f-g)(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$ 5

(b) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Show that the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable in (a, b) and that $F'(c) = f(c)$ for every $c \in (a, b)$. 5

20. (a) Find all values of $n \in \mathbb{N}$ for which the improper integral $\int_a^b \frac{dx}{(b-x)^n}$ converges. 5

(b) Show that

$$\Gamma(n+1) = n!\ \forall\ n \in \mathbb{N}$$ 5

21. (a) Show that $\int_0^{\pi/2} \log \sin x\,dx$ is convergent and find its value. 3+2=5

(b) Show that the gamma integral

$$\int_0^\infty x^{n-1}e^{-x}\,dx$$

converges for $n > 0$. 5

22. (a) Show that if a sequence of functions converges uniformly to some function, then it converges pointwise to the same function. Give an example to justify that the converse is not true. 2+3=5

(b) Test the sequence

$$\left\{\frac{nx}{1+n^2x^2}\right\}, -1 \le x \le 1, n \in \mathbb{N}$$

for uniform convergence. 5

23. (a) Let $f_n \to f$ pointwise on $[a, b]$ and

$$M_n = \sup\{|f_n(x) - f(x)| : x \in [a, b]\}$$

Show that $f_n \to f$ uniformly iff $M_n \to 0$ as $n \to \infty$. 5

(b) Let $f_n \to f$ uniformly in $[a, b]$ and each f_n is continuous in $[a, b]$. Is f necessarily continuous in $[a, b]$? Justify. 5

24. (a) Let $\{a_n\}$ be a bounded sequence of real numbers and $\varepsilon > 0$. Show that there exists $k \in \mathbb{N}$ such that $a_n > a - \varepsilon \; \forall \; n \ge k$, where $a := \liminf\limits_{n} a_n$. 5

(b) Find the radius of convergence for the following power series : 3+2=5

(i) $x + \frac{x^2}{2^2} + \frac{2!x^3}{3^3} + \frac{3!x^4}{4^4} + \cdots$

(ii) $\sum\limits_{n=0}^{\infty} \frac{x^n}{n^n}$

25. (a) Find the limit inferior and limit superior for the sequence $\{a_n\}$, where

$$a_n = \sin\frac{n\pi}{3}, \; n \in \mathbb{N}$$ 5

(b) State and prove Cauchy-Hadamard formula for radius of convergence of power series. 5

★★★