2022/TDC (CBCS)/EVEN/SEM/ MTMHCC–402T/260

TDC (CBCS) Even Semester Exam., 2022

MATHEMATICS

(Honours)

(4th Semester)

Course No. : MTMHCC–402T

(Riemann Integration and Series of Functions)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following questions : 2×10=20

1. Write two partitions of the closed interval [0, 1] such that one is a refinement of the other.

2. If P is a refinement of Q, where P and Q are partitions of $[a, b]$, what is the relation among $L(P, f)$, $L(Q, f)$, $U(P, f)$ and $U(Q, f)$?

3. Give example, with justification, of a function that is not Riemann integrable in [0, 1].

4. Evaluate :

$$\int_1^3 |x-2|\,dx$$

5. If f and g are functions on $[a, b]$ such that f is integrable and g is not integrable, can $f+g$ be integrable? Justify.

6. Let

$$f(x) = \frac{1}{x^2+1}, \quad \forall x \in [-1, 1]$$

Find a point $c \in [-1, 1]$ such that

$$f(c) = \int_{-1}^{1} \frac{dx}{1+x^2}$$

7. Test the convergence of

$$\int_0^1 \frac{\sin x}{x^3}\,dx$$

8. Evaluate :

$$\int_0^1 \frac{dx}{\sqrt{1-x}}$$

9. Evaluate :

$$\int_1^{\infty} \frac{dx}{x^2(x+1)}$$

10. Define pointwise and uniform convergence of a sequence of functions defined on $A \subseteq \mathbb{R}$.

11. Find the limit function of the sequence of functions $< f_n >$, where

$$f_n(x) = \frac{1}{1+nx}, \quad x \in [0, 1]$$

12. Use Weierstrass M-test to test the convergence of Σf_n, where

$$f_n(x) = \frac{1}{n^2+x^2}, \quad x \in \mathbb{R}$$

13. Find the limit superior and limit inferior of $< x_n >$, where

$$x_n = 1 + (-1)^n \quad \forall n \in \mathbb{N}$$

14. What is the radius of convergence and the interval of convergence of the power series

$$\sum_{n=1}^{\infty} 2^n x^n ?$$

15. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(n+1)x^n}{(n+2)(n+3)}$$

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. *(a)* Let $f:[a, b] \to \mathbb{R}$ be a bounded function. Show that f is Darboux integrable iff for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$. 5

(b) Show that $f:[0, 1] \to \mathbb{R}$ defined by

$$f(x) = x, \forall x \in \mathbb{R}$$

is Darboux integrable. 5

17. *(a)* If $f:[a, b] \to \mathbb{R}$ is monotone, then show that f is integrable. 5

(b) If f is integrable on $[a, b]$, then show that $|f|$ is also intgrable and

$$\left|\int_a^b f\, dx\right| \le \int_a^b |f|\, dx$$ 5

18. *(a)* If f and g are integrable on $[a, b]$, then show that $f+g$ is also integrable on $[a, b]$ and

$$\int_a^b (f+g)\, dx = \int_a^b f\, dx + \int_a^b g\, dx$$ 4+2=6

(b) If f is continuous on $[a, b]$, then show that $\exists\, \xi \in [a, b]$ such that

$$\int_a^b f\, dx = f(\xi)(b-a)$$ 4

19. *(a)* State and prove the fundamental theorem of integral calculus in any one of the two forms. 1+5=6

(b) Let f be integrable on $[a, b]$, and $c \in [a, b]$. Show that

$$\int_a^b f\, dx = \int_a^c f\, dx + \int_c^b f\, dx$$ 4

20. *(a)* Show that

$$\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

converges if and only if both m and n are positive. 5

(b) Evaluate : 5

$$\int_0^1 \sqrt{1-x^4}\, dx$$

21. *(a)* Using comparison test, examine the convergence of

$$\int_0^\infty \frac{dx}{1+x^3}$$

Also evaluate the value of the integral. 2+4=6

(b) Prove that

$$\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)$$ 4

22. *(a)* Let $< f_n >$ be a sequence of continuous functions defined on $A \subseteq \mathbb{R}$ and $< f_n >$ converges uniformly to a function $f : A \to \mathbb{R}$. Show that f is also continuous on A. 5

(b) Test the uniform convergence of—

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(x^2 + n^2 x)}{n(n+1)}$;

(ii) $\displaystyle\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \cdots$

where $-\frac{1}{2} \le x \le \frac{1}{2}$. 2+3=5

23. *(a)* Let $< f_n >$ be a sequence of functions on $[a, b]$ converging pointwise to a function $f : [a, b] \to \mathbb{R}$. Let

$$M_n = \sup\{|f_n(x) - f(x)| : x \in [a, b]\}$$

Show that $< f_n >$ converges to f uniformly iff $M_n \to 0$ as $n \to \infty$. 5

(b) Let

$$f_n(x) = \frac{\sin nx}{1+nx}, \quad x \ge 0$$

Show that $< f_n >$ converges uniformly on any interval $[a, \infty)$ where $a > 0$ but fails to converge uniformly on $[0, \infty)$. 5

24. *(a)* Show that if a power series

$$\sum a_n x^n$$

converges for $x = x_0$, then it is absolutely convergent for every $x = x_1$ where $|x_1| < |x_0|$. 5

(b) Find the radius of convergence R of the power series

$$\sum \frac{x^{2n}}{3^n}$$

Hence find the values of x for which the series converges. Discuss the special case of $|x| = \pm R$. 5

25. *(a)* Show that if a power series

$$\sum a_n x^n$$

converges for $|x| < R$, then it converges uniformly on $[-R+\varepsilon, R-\varepsilon]$ for every $\varepsilon > 0$. 4

(b) Find the radii of convergence of the following power series : 3+3=6

(i) $\frac{1}{2}x + \frac{1.3}{2.5}x^2 + \frac{1.3.5}{2.5.8}x^3 + \cdots$

(ii) $x + \frac{x^2}{2^2} + \frac{2!x^3}{3^3} + \frac{3!x^4}{4^4} + \cdots$

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