2021/TDC/CBCS/ODD/
MATHCC-302T/326

**TDC (CBCS) Odd Semester Exam., 2021**
**held in March, 2022**

# MATHEMATICS

**( 3rd Semester )**

Course No. : MATHCC-302T

**( Group Theory )**

*Full Marks* : 70
*Pass Marks* : 28

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

## SECTION—A

Answer any *ten* of the following questions : 2×10=20

**1.** Define quaternion group.

**2.** If G is a group, then show that

$$(xy)^{-1} = y^{-1}x^{-1}\ \forall\ x, y \in G$$

3. Show that identity element in a group is unique.

4. Define centre of a group.

5. Prove that a non-empty subset $H$ of a group $G$ is a subgroup of $G$ iff $HH^{-1} = H$.

6. What do you mean by product of two subgroups?

7. Find the generators of the group $G = \{1, -1, i, -i\}$ under the operation multiplication.

8. Is the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

a transposition?

9. When is an additive group $G$ said to be cyclic?

10. Define factor group.

11. State Fermat's little theorem.

12. If $H$ is a subgroup of a group $G$, then show that $aH = bH$ iff $a^{-1}b \in H$.





13. What do you mean by group homomorphism?

14. If $f : G \to G'$ is an isomorphism, what is the Kernel of $f$?

15. Let $f : G \to G'$ be a homomorphism. Show that $f(G)$ is a subgroup of $G'$.

## SECTION—B

Answer any *five* of the following questions : 10×5=50

16. (a) Show that

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R, ad - bc \neq 0 \right\}$$

is a group with respect to matrix multiplication. Is this group abelian? 5

(b) Prove that if $G$ is an abelian group, then for all $a, b \in G$ and all integers $n$, $(ab)^n = a^n b^n$. 5

17. (a) (i) Define order of a group.

(ii) If in the group $G$, $a^5 = e$, $aba^{-1} = b^2$ for $a, b \in G$, find $O(b)$. 2+4=6

(b) Let $G$ be a group and $a, b \in G$. Then show that the equation $ax = b$ has unique solution in $G$. 4

**18.** *(a)* Let $H$ and $K$ be any two subgroups of a group $G$. Then show that $HK$ is a subgroup of $G$ iff $HK = KH$.

*(b)* Define order of an element of a subgroup. Show that a non-empty subset $H$ of a finite group $G$ is a subgroup of $G$ iff $HH = H$. 2+3=5

**19.** *(a)* What do you mean by normalizer of a group? Show that the intersection of arbitrary collection of subgroups of a group is a subgroup of the group. 2+3=5

*(b)* Prove that the set

$$H = \{x \in G \mid gx = xg,\ g \in G\}$$

is a subgroup of a group $G$. Also prove that $G$ is abelian $\Leftrightarrow G = H$. 2½+2½=5

**20.** *(a)* Prove that a group of order $n$ is cyclic iff it has an element of order $n$. 5

*(b)* What do you mean by composition of two permutations? If $f = (1\ 3\ 5)$ and $g = (2\ 6\ 7)$ be two disjoint cycles on a set $S$ having 7 elements, check whether $f \circ g = g \circ f$ or not. 2+3=5





**21.** *(a)* Define alternating group, even permutation and odd permutation. 2+2+2=6

*(b)* When is a group said to be abelian? Show that every cyclic group is abelian. 1+3=4

**22.** *(a)* What do you mean by right coset of a subgroup of a group? Prove that any two right cosets are either disjoint or identical. 2+3=5

*(b)* Define normal subgroup of a group. Prove that a subgroup $H$ of a group $G$ is normal iff $gHg^{-1} = H\ \forall g \in G$. 1+4=5

**23.** *(a)* What do you mean by index of a subgroup? Show that if $H$ is a subgroup of a group $G$, such that $i_G(H) = 2$, then $H$ is normal in $G$. 2+3=5

*(b)* State and prove Lagrange's theorem. 1+4=5

**24.** *(a)* State and prove Cayley's theorem. 5

*(b)* Show that if $G$ be a cyclic group, then the automorphism of $G$ is abelian. 5

25. *(a)* State and prove fundamental theorem of homomorphism. 6

*(b)* Show that union of two normal subgroups may not be a normal subgroup. 4

★★★