2022/TDC/ODD/SEM/
MTMHCC–302T/326

TDC (CBCS) Odd Semester Exam., 2022

MATHEMATICS

(Honours)

(3rd Semester)

Course No. : MTMHCC–302T

(Group Theory)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* questions of the following : 2×2=4

(a) Write the order of the group of symmetries of a square. Is this group Abelian? 1+1=2

(b) Write down the elements of the group S_4, the permutation group on $\{1, 2, 3, 4\}$. 2

(c) Give one example of an Abelian group of order 8 and one non-Abelian group of order 8. 1+1=2

2. Answer *either* (a) and (b) *or* (c) and (d) : 5×2=10

(a) Let G be any Abelian group and $a, b \in G$. Show that $(ab)^2 = a^2b^2$. Give an example to show that the result is not true if G is non-Abelian. 2+3=5

(b) Show that the set $\mathbb{Z}_n := \{0, 1, 2, \cdots, n-1\}$ forms a group under addition modulo n. Does $\mathbb{Z}_n$ form a group under multiplication modulo n? Justify. 3+2=5

(c) Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ $\forall a, b \in G$. 5

(d) Let $G = \{0, 1, 2\}$ and define

$$a * b = |a - b| \quad \forall a, b \in G$$

Construct the composition table for G and examine if G forms a group under $*$. 3+2=5

UNIT—II

3. Answer any *two* questions of the following : 2×2=4

(a) Find the normaliser of $H = \{\pm 1\}$ in $G = \{\pm 1, \pm i\}$.

(b) Find the centre of S_3.

(c) Prove or disprove : If H and K are subgroups of G, then HK is also a subgroup of G.

4. Answer either (a) and (b) or (c) and (d) : 5×2=10

(a) Show that centre of a group is a subgroup of the group. 5

(b) Show that intersection of two subgroups of a group is a subgroup of the group. Give an example to show that union of subgroups need not be a subgroup. 3+2=5

(c) If H and K are subgroups of G, show that HK is a subgroup of G iff $HK = KH$. 5

(d) Define centraliser of a subgroup in a group. Show that centraliser of any subgroup is again a subgroup. Establish the relation between centraliser and normaliser of a subgroup. 1+2+2=5

UNIT—III

5. Answer any *two* questions of the following : 2×2=4

(a) Give an example of cyclic group of order 6 and a non-cyclic group of order 6.

(b) Show that any prime order group is cyclic.

(c) Write down the elements of the alternating group A_4 on {1, 2, 3, 4}.

6. Answer either (a) and (b) or (c) and (d) : 5×2=10

(a) Show that every subgroup of a cyclic group is cyclic.

(b) Prove that disjoint cycles in S_n commute with each other.

(c) Show that A_n is a subgroup of S_n and $O(A_n) = \frac{1}{2} \cdot n!$.

(d) Let G be a finite cyclic group of order n. For $d > 0$, $d|n$, show that G has a unique subgroup of order d.

UNIT—IV

7. Answer any *two* questions of the following : 2×2=4

(a) Can a group of order 12 have a subgroup of order 8? Justify.

(b) Find the unit digit in 9^{223}.

(c) Give an example of a normal subgroup of S_3 and another subgroup of S_3 which is not normal in S_3.

8. Answer either (a) and (b) or (c) and (d) : 5×2=10

(a) State and prove Lagrange's theorem. 5

(b) State Fermat's little theorem. Using it or otherwise, find the remainder when 4^{293} is divided by 17. 2+3=5

(c) Give a proof of Fermat's little theorem using group theory. 5

(d) Give examples of 5 distinct groups of order 8. 5

UNIT—V

9. Answer any *two* questions of the following : 2×2=4

(a) Are the groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ isomorphic? Justify.

(b) Give an isomorphism between the group of real numbers under addition and the group of positive real numbers under multiplication.

(c) Show that every normal subgroup of a group must be the kernel of some homomorphism from the group.

10. Answer either *(a)* and *(b)* or *(c)* and *(d)* : 5×2=10

(a) State and prove Cayley's theorem. 1+4=5

(b) Show that the kernel of a group homomorphism is a normal subgroup of the group. 5

(c) State and prove first theorem of group isomorphism. 1+4=5

(d) Find all group homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_6$. 5

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