2020/TDC (CBCS)/ODD/SEM/ MTMHCC-302T/328

TDC (CBCS) Odd Semester Exam., 2020 held in March, 2021

MATHEMATICS

(3rd Semester)

Course No. : MTMHCC-302T

(Group Theory)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

1. Answer any *ten* of the following questions : 2×10=20

   (a) Define binary operation on a set. Give an example of a binary operation.

   (b) Is $\{-1, 0, 1\}$ a group under addition operation? Justify your answer.

(c) Define a quaternion group.

(d) If G be a group, then show that

$$ab = ac \Rightarrow b = c \quad a, b, c \in G$$

(e) Define product of two subgroups of a group. Give an example of a subgroup.

(f) What do you mean by the centre of a group?

(g) Is the union of two subgroups of a group a subgroup of the group? Justify your answer.

(h) If H is any subgroup of a group G, then prove that $H^{-1} = H$. Is the converse true?

(i) Define a cyclic group. Give one example.

(j) If a is a generator of a cyclic group G, then prove that a^{-1} is also a generator of G.

(k) Define alternating group.

(l) When is a permutation said to be even or odd permutation?

(m) What do you mean by the external direct product of two groups?

(n) Define factor group.

(o) Is the converse of Lagrange's theorem true? Justify your answer.

(p) When is a left coset equal to the corresponding right coset?

(q) When are two groups said to be isomorphic?

(r) Let G and G' be two groups and $f : G \to G'$ be an isomorphism. If $e \in G$ is the identity element of G, then show that $f(e) \in G'$ is the identity element of G'.

(s) Show that the relation of isomorphism in the set of all groups is reflexive and symmetric.

(t) If $G \to G'$ is an isomorphism, then show that the order of an element a of G is equal to the order of its image $f(a)$.

SECTION—B

Answer *any* **five** questions

2. (a) If

$$G = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \text{ is any non-zero real number} \right\}$$

then show that G is a commutative group under matrix multiplication. 5

(b) Show that the set P_n of all permutations on n symbols is a finite group of order n with respect to composite of mappings as operation. Also, show that for $n \leq 2$, the group is Abelian and for $n > 2$, it is always non-Abelian. 5

3. (a) Prove that the set of all n-th roots of unity forms a finite Abelian group of order n with respect to the operation of multiplication. 4

(b) Show that multiplication of permutations is not commutative, in general. 3

(c) If a and b are two elements of a group G, then show that the equations $ax = b$ and $ya = b$ have unique solutions in G. 3

4. (a) If H and K are two subgroups of a group G, then show that HK is a subgroup of G iff $HK = KH$. 5

(b) Define normalizer of an element of a group. Show that it is a subgroup of the group. 1+4=5

5. (a) Show that the union of two subgroups of a group is a subgroup iff one is contained in the other. Give example of two subgroups whose union is not a subgroup. 4+1=5

(b) Prove that a non-empty subset H of a group G is a subgroup of G iff $a, b \in H \Rightarrow ab \in H$ and $a \in H \Rightarrow a^{-1} \in H$, where a^{-1} is the inverse of a in G. 5

6. (a) Define cyclic permutation. Write the following permutation as the product of disjoint cycles :

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 1 & 6 & 7 & 9 & 8 \end{pmatrix}$$

Is it an even permutation? 1+2+1=4

(b) How many generators are there in a cyclic group of order 8? 3

(c) Give an example of a finite Abelian group which is not cyclic. 3

7. (a) Prove that every group of prime order is cyclic. 3

(b) Define identity permutation. Find the inverse of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$ 1+2=3

(c) Show that every subgroup of a cyclic group is cyclic. 4

8. (a) State and prove Lagrange's theorem. 1+4=5

(b) Prove that a subgroup H of a group G is a normal subgroup iff the product of two right cosets of H in G is again a right coset of H in G. 5

9. (a) Show that any two left cosets of a subgroup of a group is either disjoint or identical. 3

(b) If H is a subgroup of a group G and $h \in H$, then show that $Hh = H = hH$. 3

(c) If H is a subgroup of a group G and N is a normal subgroup of G, then show that $H \cap N$ is a normal subgroup of H. 4

10. (a) State and prove Cayley's theorem. 1+4=5

(b) If $\mathbb{R}$ is the additive group of real numbers and $\mathbb{R}_+$ is the multiplicative group of positive real numbers, then prove that the mapping $f : \mathbb{R} \to \mathbb{R}_+$ defined by $f(x) = e^x \; \forall x \in \mathbb{R}$ is an isomorphism of $\mathbb{R}$ onto $\mathbb{R}_+$. 5

11. (a) State and prove the first theorem on isomorphism. 1+4=5

(b) Define group homomorphism. If f is an isomorphism from a group G onto a group G', then show that f^{-1} is also an isomorphism from G' onto G. 1+4=5

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