2019/TDC/ODD/SEM/MTMHCC-302T/176

TDC (CBCS) Odd Semester Exam., 2019

MATHEMATICS

(3rd Semester)

Course No. : MTMHCC–302T

(Group Theory)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following questions : 2×2=4

(a) Define group with example.

(b) What do you mean by the order of an element of a group? What is the order of an infinite group?

(c) Define a semigroup. Is it a group?

2. Answer either [(a) and (b)] or [(c) and (d)] :

(a) Show that the set M of all complex numbers z such that $|z| = 1$ form a group w.r.t. the operation of multiplication of complex numbers. Is it abelian? 4+1=5

(b) Consider the permutations

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

Compute $f * g$, $g * f$ and f^{-1}. 2+2+1=5

(c) Prove that the set of all nth roots of unity forms a finite abelian group of order n w.r.t. multiplication. 5

(d) Define symmetric group. Show that the symmetric group S_3 is non-abelian. What is the order of S_n? 1+3+1=5

UNIT—II

3. Answer any *two* of the following questions : 2×2=4

(a) What is the index of a subgroup of a group?

(b) Write two subgroups of $\mathbb{Z}$ under addition.

(c) Is the union of two subgroups a subgroup? Justify your answer.

4. Answer either [(a) and (b)] or [(c) and (d)] :

(a) Define inverse of a complex. If H is a subgroup of a group G, then show that $H^{-1} = H$. Also show that the converse is not true. 1+2+2=5

(b) What do you mean by the product of two subgroups of a group? Show that the product of two subgroups H and K of a group G is a subgroup iff $HK = KH$. 1+4=5

(c) What is the normalizer of an element of a group? Prove that the centre of a group G is a subgroup of G. 2+3=5

(d) Prove that the necessary and sufficient condition for a nonempty subset H of a group G to be a subgroup is that $ab^{-1} \in H$, where $a, b \in H$ and b^{-1} is the inverse of b in G. 5

UNIT—III

5. Answer any *two* of the following questions : 2×2=4

(a) What are the generators of the cyclic group $\{1, -1, i, -i\}$?

(b) Define alternating group. What is its order?

(c) What is the length of an identity permutation? Is it cyclic?

6. Answer either [(a) and (b)] or [(c) and (d)] :

(a) Define a cyclic group. How many generators are there of a cyclic group of order 8? 2+3=5

(b) When are two cyclic permutations said to be disjoint? Give an example to show that the product of two disjoint cyclic permutations on a set commute with each other. 2+3=5

(c) Prove that—

(i) every group of prime order is cyclic;

(ii) if a is a generator of a cyclic group G, then a^{-1} is also a generator of G. 3+2=5

(d) What do you mean by even and odd permutations? Give one example of each. Is the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ odd or even? 2+2+1=5

UNIT—IV

7. Answer any *two* of the following questions : 2×2=4

(a) Define a factor group.

(b) Define normal subgroup with example.

(c) Define simple group and give an example of it.

8. Answer either [(a) and (b)] or [(c) and (d)] :

(a) State and prove Lagrange's theorem. 1+4=5

(b) (i) Show that every subgroup of a cyclic group is normal. 3

(ii) Show that the factor group of an abelian group is abelian. 2

(c) Define right coset and left coset of a subgroup of a group. When are they same? Show that any two right cosets are either disjoint or identical. 1+1+3=5

(d) If H and K are two subgroups of a group G and H is normal in G, then prove that HK is a subgroup of G and $H \cap K$ is a normal subgroup of K. 5

UNIT—V

9. Answer any *two* of the following questions : 2×2=4

(a) What do you mean by group homomorphism?

(b) Show that the homomorphic image of an abelian group is abelian.

(c) Let G and G' be two groups and $f : G \to G'$ be a homomorphism. Then show that $f(a^{-1}) = [f(a)]^{-1}$, $\forall\, a \in G$.

10. Answer either [*(a)* and *(b)*] or [*(c)* and *(d)*] :

(a) Define kernel of a homomorphism. If $f : G \to G'$ be a homomorphism, then show that kernel of f is a normal subgroup of G. 1+4=5

(b) State and prove Cayley's theorem. 1+4=5

(c) Write down the identity element of a quotient group. Show that any infinite cyclic group is isomorphic to the group of integers under addition. 1+4=5

(d) State and prove the fundamental theorem of homomorphism. 1+4=5

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