2022/TDC/ODD/SEM/
MTMHCC–301T/325

TDC (CBCS) Odd Semester Exam., 2022

MATHEMATICS

(Honours)

(3rd Semester)

Course No. : MTMHCC–301T

(Theory of Real Functions)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following questions : 2×2=4

(a) Show that

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$

does not exist in $\mathbb{R}$.

(b) Using squeeze theorem, show that

$$\lim_{x\to 0}\left(\frac{\cos x-1}{x}\right)=0$$

(c) Give examples of functions f and g such that f and g do not have limits at a point c, but such that both $f+g$ and fg have limits at c.

Answer *either* Q.No. **2.** *or* **3.** :

2. *(a)* Let $A(\neq \phi)\subseteq \mathbb{R}$, let $f: A\to\mathbb{R}$ and let $c\in\mathbb{R}$ be a cluster point of A. If $\lim_{x\to c} f$ exists, and if $|f|$ denotes the function defined for $x\in A$ by $|f|(x):=|f(x)|$, prove that

$$\lim_{x\to c}|f|=\left|\lim_{x\to c} f\right|$$ 5

(b) Let $f: A\to\mathbb{R}$ and let c be a cluster point of A. Then prove that the following are equivalent : 5

(i) $\lim_{x\to c} f = L$

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n\in\mathbb{N}$, the sequence $(f(x_n))$ converges to L

3. *(a)* Let $A\subseteq R$, let $f, g, h: A\to\mathbb{R}$ and let $c\in\mathbb{R}$ be a cluster point of A. If $f(x)\le g(x)\le h(x)$, for all $x\in A$, $x\neq c$ and if

$$\lim_{x\to c} f = L = \lim_{x\to c} h$$

then prove that

$$\lim_{x\to c} g = L$$ 5

(b) Let $\phi\neq A\subseteq\mathbb{R}$, let $f, g: A\to\mathbb{R}$ and let $c\in\mathbb{R}$ be a cluster point of A. Suppose that $f(x)\le g(x)$ for all $x\in A$, $x\neq c$. Prove that

(i) if $\lim_{x\to c} f=\infty$, then $\lim_{x\to c} g=\infty$

(ii) if $\lim_{x\to c} g=-\infty$, then $\lim_{x\to c} f=-\infty$ 5

UNIT—II

4. Answer any *two* of the following questions : 2×2=4

(a) Give examples of two functions f and g on $\mathbb{R}$ such that f is continuous at every point of $\mathbb{R}$ and g is not continuous at any point of $\mathbb{R}$.

(b) Give an example of a function $f:[0, 1]\to\mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$.

(c) Let $F : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$F(x) := \begin{cases} 5 & , \text{ for } x = 0 \\ x \sin\left(\frac{5}{x}\right), & \text{for } x \neq 0 \end{cases}$$

Show that F is not continuous at $x = 0$.

Answer *either* Q.No. **5.** *or* **6.** :

5. (a) Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$. Suppose that $c \in A$ and that f and g are continuous at c. Prove that $f + g$ and fg are continuous at c. 5

(b) Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Prove that f has an absolute maximum and an absolute minimum on I. 5

6. (a) Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $k \in \mathbb{R}$ is any number satisfying

$$\inf f(I) \leq k \leq \sup f(I)$$

then prove that there exists a number $c \in I$ such that $f(c) = k$. 5

(b) Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. Prove that the set $f(I)$ is an interval. 5

UNIT—III

7. Answer any *two* of the following questions : 2×2=4

(a) If $f : I \to R$ has a derivative at $c \in I$, then prove that f is continuous at c.

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at c and that $f(c) = 0$. Show that $g(x) = |f(x)|$ is differentiable at c if $f'(c) = 0$.

(c) Prove that $e^x \geq 1 + x$ for $x \in \mathbb{R}$.

Answer *either* Q.No. **8.** *or* **9.** :

8. (a) Let $I \subseteq R$ be an interval, let $c \in I$ and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at c. Prove that the functions αf and $f + g$ are differentiable at c and

$$(\alpha f)'(c) = \alpha f'(c) \text{ and } (f + g)'(c) = f'(c) + g'(c)$$

where $\alpha \in \mathbb{R}$. 5

(b) State and prove Caratheodory's theorem. 5

9. (a) Suppose that f is continuous on a closed interval $I = [a, b]$ and that f has a derivative in the open interval (a, b). Then prove that there exists at least one point c in (a, b) such that

$$f(b) - f(a) = f'(c)(b - a)$$ 5

(b) State and prove Darboux's theorem. 5

UNIT—IV

10. Answer any *two* of the following questions : 2×2=4

(a) Show that the function $f(x) = \frac{1}{x}$ is uniformly continuous on the set $A = [a, \infty)$, where a is a positive constant.

(b) If f is uniformly continuous on $A \subseteq R$ and $|f(x)| \geq k > 0$ for all $x \in A$, show that $\frac{1}{f}$ is uniformly continuous on A.

(c) Define Lipschitz function and give an example.

Answer *either* Q.No. **11.** *or* **12.** :

11. (a) Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then prove that f is uniformly continuous on I. 5

(b) Show that if f is continuous on $[0, \infty)$ and uniformly continuous on $[a, \infty)$, for some positive constant a, then f is uniformly continuous on $[0, \infty)$. 5

12. (a) Show that the function $f : (0, 1) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is uniformly continuous. 5

(b) If $f : A \to R$ is a Lipschitz function, then prove that f is uniformly continuous on A. Give an example with justification that a uniformly continuous function may not be a Lipschitz function. 3+2=5

UNIT—V

13. Answer any *two* of the following questions : 2×2=4

(a) Define a convex function and give an example.

(b) Show that

$$\frac{v - u}{1 + v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v - u}{1 + u^2}$$

if $0 < u < v$.

(c) Expand $\sin x$ in ascending powers of x.

Answer *either* Q.No. **14.** *or* **15.** :

14. (a) State and prove Taylor's theorem with the Lagrange's form of remainder. 5

(b) Show that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

for $-1 < x \le 1$. 5

15. (a) State and prove Cauchy's mean value theorem. 5

(b) Prove or disprove : For every differentiable function $f : \mathbb{R} \to \mathbb{R}$ and for every $x_0 \in \mathbb{R}$ there exist $a, b \in \mathbb{R}$ with $a < x_0 < b$ such that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$ 5

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