2020/TDC (CBCS)/ODD/SEM/ MTMHCC–301T/327

TDC (CBCS) Odd Semester Exam., 2020 held in March, 2021

MATHEMATICS

(3rd Semester)

Course No. : MTMHCC–301T

(Theory of Real Functions)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

1. Answer any *ten* of the following questions : 2×10=20

(a) Using $\varepsilon-\delta$ definition, prove that

$$\underset{x\to 2}{\text{Lt}}\,(5x+7)=17$$

(b) Show that

$$\underset{x\to 0}{\text{Lt}}\ \sin\frac{1}{x}$$

does not exist.

(c) Show that

$$\underset{x\to 0}{\text{Lt}} \sin\frac{1}{x^2} = \infty$$

(d) Define limits at infinity.

(e) State the sequential criterion for continuity of a function.

(f) Examine the continuity of the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}1, & \text{if } x \text{ is rational}\\ -1, & \text{if } x \text{ is irrational}\end{cases}$$

at every rational point in $\mathbb{R}$.

(g) How can you 'remove' the discontinuity of the function

$$f(x)=\begin{cases}x+1, & x>1\\ 1, & x=1\\ 3-x, & x<1\end{cases}$$

at $x=1$?

(h) Check if there exists a zero of $f(x)=x^3-x-2$, not exceeding 2.

(i) Check the differentiability of $f(x)=|x+a|\ \forall x\in\mathbb{R}$ at $x=-a$.

(j) State Rolle's theorem.

(k) Use definition to find the derivative of the function $f(x)=\sqrt{x},\ x>0$.

(l) Show that differentiability of a function $f:\mathbb{R}\to\mathbb{R}$ implies continuity of f.

(m) Show that the function $f:(-2, 2)\to\mathbb{R}$, defined by $f(x)=x^3\ \forall\ x\in(-2, 2)$ is uniformly continuous in $(-2, 2)$.

(n) Show that the function $f:(0,\infty)\to\mathbb{R}$ defined by $f(x)=\frac{1}{x}\ \forall x\in(0,\infty)$ is not uniformly continuous.

(o) What is a Lipschitz function? Explain with example.

(p) Mention the non-uniform continuity criterion using sequences.

(q) Using Lagrange's mean value theorem, show that $e^x\geq 1+x\ \ \forall\ x\in\mathbb{R}$.

(r) State Taylor's theorem.

(s) Write the Taylor expansion of $f(x) = \sin x$ at $x = 0$.

(t) Give an example of a function which is strictly monotone increasing but not continuous.

SECTION—B

Answer any *five* of the following questions : 10×5=50

2. (a) Using $\varepsilon - \delta$ definition, show that

$$\underset{x \to c}{\text{Lt}} \frac{1}{x} = \frac{1}{c}, \quad c > 0$$ 5

(b) Prove the equivalence of $\varepsilon - \delta$ definition and sequential definition of limit of a function at a point. 5

3. (a) Using $\varepsilon - \delta$ definition, show that

$$\underset{x \to 1}{\text{Lt}} \frac{x}{1+x} = \frac{1}{2}$$ 5

(b) If $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then show that f is bounded on some neighbourhood of c. 5

4. (a) Show that a function continuous on a closed and bounded interval $[a, b]$ is itself bounded therein. 5

(b) If f and g are continuous at $x = c$, show that the product function fg is continuous at $x = c$. 5

5. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let I be an interval. Show that $f(I)$ is an interval. 5

(b) Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(r) = 0$ for every rational number r. 5

6. (a) If f and g are differentiable at $x = c$ and $g(c) \neq 0$, then show that the quotient f / g is differentiable at $x = c$. 5

(b) State and prove Lagrange's mean value theorem. 1+4=5

7. (a) Let $f : I \to \mathbb{R}$ be differentiable on the interval I. Then show that f is increasing on I iff $f'(x) \geq 0 \ \forall \ x \in I$. 5

(b) State and prove Darboux's theorem. 1+4=5

8. *(a)* Show that if f and g are uniformly continuous on $A \subseteq \mathbb{R}$, then $f+g$ is uniformly continuous on A. 5

(b) If f is uniformly continuous on $A \subseteq \mathbb{R}$ and $|f(x)| \geq k > 0 \ \forall \ x \in A$, show that $1/f$ is uniformly continuous on A. 5

9. *(a)* Every Lipschitz function is uniformly continuous. Can we also say that every uniformly continuous function is Lipschitz? Justify your answer. 5

(b) Show that $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{1+x^2} \ \forall \ x \in \mathbb{R}$$

is uniformly continuous. 5

10. *(a)* State and prove Cauchy's mean value theorem. 5

(b) Using Taylor's theorem, show that

$$1 - \frac{x^2}{2} \leq \cos x \quad \forall \ x \in \mathbb{R}$$ 5

11. *(a)* Derive the Taylor's series expansion of $f(x) = \ln(1+x)$ about any point $x = c$ in its domain. 5

(b) Show that if $x > 0$, then

$$1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}$$ 5

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