2021/TDC/CBCS/ODD/
MATHCC–301T/325

TDC (CBCS) Odd Semester Exam., 2021
held in March, 2022

MATHEMATICS

(3rd Semester)

Course No. : MATHCC–301T

(Theory of Real Functions)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following questions : 2×10=20

1. Show that $\lim_{x \to 0} \frac{1}{x}$, $x > 0$ does not exist.

2. Using squeeze theorem, show that

$$\lim_{x \to 0} \left(x \sin \frac{1}{x} \right) = 0$$

3. Show that $\lim_{x \to 0} \frac{1}{x^2} = \infty$

4. State the sequential criterion for continuity.

5. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is not continuous at any point of $\mathbb{R}$.

6. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$, $x \in \mathbb{R}$ is continuous at every point of $\mathbb{R}$.

7. Let $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ has a derivative at $c \in I$. Then prove that f is continuous at c.

8. Show that the function

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$$

is differentiable at all $x \in \mathbb{R}$.

9. Show that $-x \leq \sin x \leq x$, for all $x \geq 0$.

10. Define uniformly continuous function and give an example.

11. Show that if a function f is uniformly continuous on $A \subseteq R$, then it is continuous on A.

12. Define Lipschitz function and give an example.

13. Show that $\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$, where $0 < \alpha < \theta < \beta < \frac{\pi}{2}$ by using Cauchy's mean value theorem.

14. Define a convex function.

15. Expand $\cos x$ in ascending powers of x.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. (a) Let $A \subseteq R$, $c \in \mathbb{R}$ is a cluster point of A and $f : A \to R$ be a function. Define limit of f at c. Further show that if $f : A \to R$ and c is a cluster point of A, then f can have at the most one limit at c. 1+4=5

(b) If $A \subseteq R$ and $f : A \to R$ has limit at $c \in \mathbb{R}$, then show that f is bounded on some neighbourhood of c. 5

17 (a) Let $A \subseteq R$, f and g be functions on A to $\mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. If $\lim_{x \to c} f = L$ and $\lim_{x \to c} g = M$, then prove that

$$\lim_{x \to c} (f + g) = L + M$$ 5

(b) State squeeze theorem on limits. Using squeeze theorem, show that

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$$

2+3=5

18. (a) Let $A \subseteq R$, let $f: A \to \mathbb{R}$ and let $|f|$ be defined by $|f|(x) := |f(x)|$ for $x \in A$. Prove that if f is continuous on A, then $|f|$ is also continuous on A. Show with an example that if $|f|$ is continuous, then f may not be continuous. 5

(b) Let $A, B \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ be continuous on A, and let $g: B \to R$ be continuous on B. If $f(A) \subseteq B$, then prove that the composite function $g \circ f: A \to R$ is continuous on A. 5

19. (a) Let $I := [a, b]$ be a closed bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. Prove that f is bounded on I. 5

(b) Let I be an interval and let $f: I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then prove that there exists a point $c \in I$ between a and b such that $f(c) = k$. 5

20. (a) Let c be an interior point of an interval I at which $f: I \to R$ has a relative extremum. If the derivative of f at c exists, then prove that $f'(c) = 0$. 5

(b) Let f be a continuous function on the closed interval $I = [a, b]$ and that f is differentiable on the open interval (a, b) and that $f'(x) = 0$ for $x \in (a, b)$. Then prove that f is constant on I. 5

21. (a) State and prove Darboux's theorem. 1+4=5

(b) Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$ and let $f: I \to R$ and $g: I \to R$ be two functions that are differentiable at c. Then prove that the function fg is differentiable at c and $(fg)'(c) = f'(c)\,g(c) + f(c)\,g'(c)$. 5

22. (a) Prove that if $f: A \to R$ is a Lipschitz function, then f is uniformly continuous on A. Give an example to show that a uniformly continuous function may not be a Lipschitz function with justification. 3+2=5

(b) Prove that if $f: A \to R$ is uniformly continuous on a subset A of $\mathbb{R}$ and if (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$. 5

23. *(a)* Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then prove that f is uniformly continuous on I. 5

(b) Prove that a function f is uniformly continuous on the interval (a, b) if and only if it can be defined at the end points a and b such that the extended function is continuous on $[a, b]$. 5

24. *(a)* State and prove Cauchy's mean value theorem. 5

(b) Show that

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots \text{ for } |x| < 1$$ 5

25. *(a)* State and prove Taylor's theorem with the Lagrange form of the remainder. 5

(b) Show that $1 - \frac{1}{2}x^2 \leq \cos x$ for all $x \in \mathbb{R}$ using Taylor's theorem. 5

★ ★ ★