2019/TDC/ODD/SEM/MTMHCC–301T/175

TDC (CBCS) Odd Semester Exam., 2019

MATHEMATICS

(3rd Semester)

Course No. : MTMHCC–301T

(Theory of Real Functions)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following questions : 2×2=4

(a) If $\underset{x\to c}{\text{Lt}}\, f(x) = l$, then prove that

$$\underset{x\to c}{\text{Lt}}\, |f(x)| = |l|$$

(b) Prove that

$$\underset{x\to 1}{\text{Lt}}\, x^{\frac{1}{1-x}} = \frac{1}{e}$$

(c) State sequential criterion for limits.

2. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) If

$$\lim_{x \to 0} \frac{x^a \sin^b x}{\sin x^c}$$

where $a, b, c \in R - \{0\}$ exist and has non-zero values, then show that $a + b = c$. 3

(b) (i) If $f(x) \le g(x) \le h(x)$ in a certain neighbourhood of the point c and

$$\underset{x \to c}{\text{Lt}} f(x) = l = \underset{x \to c}{\text{Lt}} h(x)$$

then prove that

$$\underset{x \to c}{\text{Lt}} g(x) = l$$ 3

(ii) Prove that the limit of a function, if it exists, is unique. 4

(c) If

$$\underset{x \to a}{\text{Lt}} f(x) \text{ and } \underset{x \to a}{\text{Lt}} g(x)$$

exist finitely, then prove that

$$\underset{x \to a}{\text{Lt}} \{f(x) \cdot g(x)\} = \underset{x \to a}{\text{Lt}} f(x) \cdot \underset{x \to a}{\text{Lt}} g(x)$$ 4

(d) (i) Evaluate : 3

$$\underset{x \to \frac{1}{3}+}{\text{Lt}} x\left[\frac{1}{x}\right]$$

(ii) Using definition of limit, show that

$$\underset{x \to 0}{\text{Lt}} x^2 \sin\frac{1}{x} = 0$$ 3

UNIT—II

3. Answer any *two* of the following questions : 2×2=4

(a) Give an example of a bounded function which is discontinuous at every point of its domain.

(b) What must be the value of $f(0)$ so that $f(x) = (x+1)^{\cot x}$ becomes continuous at $x = 0$?

(c) Define the following terms :

(i) Removable discontinuity

(ii) Discontinuity of first kind

4. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Prove that a function f defined on an interval I is continuous at a point c in I if and only if for any sequence $\langle c_n \rangle$ in I converging to c the sequence $\langle f(c_n) \rangle$ converging to $f(c)$, i.e.,

$$c_n \to c \Rightarrow \langle f(c_n) \rangle \to f(c)$$

as $n \to \infty$. 5

(b) Show that the function

$$f(x) = |x| + |x-1| + |x-2|$$

is continuous at points $x = 0, 1, 2$. 5

(c) State and prove intermediate value theorem. 4

(d) (i) Show that the function f defined on R

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point. 3

(ii) Show that the function f defined as $f(x) = x - [x]$, where $[x]$ denotes the integral part of x is discontinuous for all integral values of x. 3

UNIT—III

5. Answer any *two* of the following : 2×2=4

(a) State Rolle's theorem.

(b) State Darboux's theorem.

(c) Let f and g be two functions with the same domain D. Give an example to show that if fg is derivable at $C \in D$, then f and g are not necessarily so at C.

6. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) State and prove Carathéodory's theorem. 4

(b) (i) Show that the function $f(x) = x|x|$ is derivable at origin. 3

(ii) Suppose f and g are continuous in $[a, b]$ and differentiable on (a, b). If $f'(x) = g'(x)\ \forall x \in (a, b)$, then prove that there exists a constant K such that $f = g + k$ on $[a, b]$, i.e., f and g differ by a constant on $[a, b]$. 3

(c) A function f is defined on R as follows :

$$f(x) = \begin{cases} \dfrac{x(e^{\frac{1}{x}} - e^{-\frac{1}{x}})}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Find $Lf'(0)$ and $Rf'(0)$. Is f derivable at $x = 0$? 4

(d) (i) If a function f is derivable at a point c and $f(c) \neq 0$, then show that function $\frac{1}{f}$ is also derivable at c and

$$\left(\frac{1}{f}\right)(c) = \frac{-f'(c)}{\{f(c)\}^2}$$ 3

(ii) Prove that, if

$$\frac{a_0}{n+1}+\frac{a_1}{n}+\cdots+\frac{a_n-1}{2+a_n}=0$$

where $a_0, a_1, \cdots, a_{n-1}, a_n$ are real numbers, then the equation

$$a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n=0$$

has at least one real root between 0 and 1. 3

UNIT—IV

7. Answer any *two* of the following : 2×2=4

(a) Define uniform continuity of a function in a domain.

(b) Prove that $f(x)=\sin x$ is uniformly continuous on R.

(c) Define Lipschitz's function.

8. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Prove that a function which is continuous in a closed and bounded interval $[a, b]$ is uniformly continuous in $[a, b]$. 5

(b) If $f: A \to R$ is uniformly continuous on a subset A on R and if $\langle x_n \rangle$ is a Cauchy sequence in A, then prove that $\langle f(x_n) \rangle$ is a Cauchy sequence in R. 5

(Continued)

(c) (i) Show that the function

$$f(x)=x^2 \ \forall x \in R$$

is uniformly continuous on $[-1, 1]$ but not in R. 3

(ii) If $f: A \to R$ is a Lipschitz function, then f is uniformly continuous on A. 3

(d) Show that the function f defined as

$$f(x)=\begin{cases}\sin\frac{1}{x}, & x \neq 0\\ 0, & x=0\end{cases}$$

is not uniformly continuous on $[0, \infty)$. 4

UNIT—V

9. Answer any *two* of the following : 2×2=4

(a) State Maclaurin's theorem with Lagrange's form of remainder.

(b) State Cauchy's mean value theorem.

(c) Suppose f is continuous on $[a, b]$ and differentiable on (a, b),

$$f'(x)=0 \ \forall x \in (a, b)$$

then show that f is constant on $[a, b]$.

(Turn Over)

10. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Using Cauchy's mean value theorem for the functions $f(x)=e^x$ and $g(x)=e^{-x}$ show that there exists a point c in (a, b) such that c is the arithmetic mean between a and b. 5

(b) State and prove Taylor's theorem with Lagrange's form of remainder. 5

(c) If f'' is continuous at $x=a$, then show that

$$\underset{h\to 0}{\text{Lt}}\left[\frac{f(a+h)-2f(a)+f(a-h)}{h^2}\right]=f''(a)$$ 4

(d) (i) Using Taylor theorem, show that

$$x-\frac{x^6}{6}<\sin x<x, \text{ for } x>0$$ 4

(ii) Explain why $\sqrt{x}$ and $x^{5/2}$ cannot be expanded in Maclaurin's infinite series. 2

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