2023/TDC(CBCS)/ODD/SEM/ MTMHCC–301T/305

TDC (CBCS) Odd Semester Exam., 2023

MATHEMATICS

(Honours)

(3rd Semester)

Course No. : MTMHCC–301T

(Theory of Real Functions)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

All the notations and terminologies have their usual meanings

SECTION—A

Answer *ten* questions, selecting any *two* from each Unit : 2×10=20

UNIT—I

1. Let $\phi \neq A \subseteq \mathbb{R}$ and let c be a cluster point of A. Define the right-hand limit and the left-hand limit of f at c.

2. Let $f(x) := \text{sgn}(x)\ \forall x \in \mathbb{R} \setminus \{0\}$. Does the limit $\lim_{x \to 0} f(x)$ exist? Justify.

3. Show that

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

UNIT—II

4. Using ε-δ definition of continuity, check if the function

$$f(x) := \begin{cases} 1, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases}$$

is continuous at 0.

5. Using ε-δ definition, define discontinuity of a function

$$f : A (\subseteq \mathbb{R}) \to \mathbb{R}$$

at some point $x_0 \in A$.

6. Prove or disprove :

If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ both are discontinuous at $x_0 \in \mathbb{R}$, then $f + g : \mathbb{R} \to \mathbb{R}$ given by

$$(f + g)(x) := f(x) + g(x) \quad \forall\, x \in \mathbb{R}$$

is also discontinuous at x_0.

UNIT—III

7. Show that a function $f : \mathbb{R} \to \mathbb{R}$ discontinuous at 0 cannot be uniformly continuous.

8. Show that every Lipschitz's continuous function is uniformly continuous.

9. Is $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \sin x \quad \forall\, x \in \mathbb{R}$$

uniformly continuous? Justify.

UNIT—IV

10. Is every continuous function

$$f : [2, \infty) \to \mathbb{R}$$

differentiable? Justify.

11. State Caratheodory's theorem on differentiablity.

12. Let $f : (0, 1) \to \mathbb{R}$ be such that

$$f(x) \neq 0 \quad \forall\, x \in (0, 1)$$

Show that f is one-one.

UNIT—V

13. State Taylor's theorem with Lagrange's form of remainder.

14. Deduce Lagrange's mean value theorem from Cauchy's mean value theorem.

15. Define a convex function $f : \mathbb{R} \to \mathbb{R}$.

SECTION—B

Answer *five* questions, selecting *one* from each Unit : 10×5=50

UNIT—I

16. *(a)* Let $A(\neq \phi) \subseteq \mathbb{R}$ and c be a cluster point of A and let $f: A \to \mathbb{R}$. Show that the following statements are equivalent : 5

(i) $\lim_{x \to c} f(x) = L$

(ii) Given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \neq c$ is any point in

$$(c-\delta,\ c+\delta) \cap A$$

then $f(x) \in (L-\varepsilon,\ L+\varepsilon)$.

(b) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by setting

$$f(x) := \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that $\lim_{x \to c} f(x)$ exists if and only if $c = 0$. 5

17. *(a)* Let $(\phi \neq) A \subseteq \mathbb{R}$, let $f, g, h: A \to \mathbb{R}$ and let c be a cluster point of A. Show that if $f(x) \leq g(x) \leq h(x)\ \forall\ x \in A,\ x \neq c$ and if $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$, then

$$\lim_{x \to c} g(x) = L$$ 5

(b) Prove that $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist, whereas

$$\lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0$$ 5

UNIT—II

18. *(a)* Show that a function

$$f: A (\subseteq \mathbb{R}) \to \mathbb{R}$$

is continuous at $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$. 5

(b) Let $I := [a, b]$ and let $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ be continuous functions on I. Show that the set

$$E := \{x \in I : f(x) = g(x)\}$$

has the property that if $(x_n) \subseteq E$ and $x_n \to x_0$, then $x_0 \in E$. 5

19. *(a)* Let $f: \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $\beta \in \mathbb{R}$. Show that if $x_0 \in \mathbb{R}$ is such that $f(x_0) < \beta$, then there exists $\delta > 0$ such that

$$f(x) < \beta\ \forall\ x \in (x_0 - \delta,\ x_0 + \delta)$$ 5

(b) Give an example, with justification, of a function $f : \mathbb{R} \to \mathbb{R}$ such that f is discontinuous at every point of $\mathbb{R}$, but $|f|$ is continuous on $\mathbb{R}$. 5

UNIT—III

20. (a) Let I, J be intervals in $\mathbb{R}$, let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be functions such that $f(J) \subseteq I$, and let $c \in J$. If f is differentiable at c and if g is differentiable at $f(c)$, then show that the composite function $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$ 5

(b) State and prove Darboux's theorem on differentiable functions. 5

21. (a) State Lagrange's mean value theorem. Use the theorem to show that

$$|\sin x - \sin y| \le |x - y| \;\forall\; x, y \in \mathbb{R}$$ 1+2=3

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) := \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Then show that—

(i) $g'(0) = 1$

(ii) given any $\delta > 0$ there exist $x_1, x_2 \in (-\delta, \delta)$ such that

$$g'(x_1)\, g'(x_2) < 0$$ 2+2=4

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) = 0$ for all $x \in \mathbb{R}$. Show that f is a constant function. 3

UNIT—IV

22. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) := x^2 \;\forall\; x \in \mathbb{R}$$

Using ε-δ definition, show that f is not uniformly continuous. 4

(b) Show that every continuous function

$$f : [0, 1] \to \mathbb{R}$$

is uniformly continuous. 6

23. (a) Check uniform continuity of $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \begin{cases} x \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$ 5

(b) Show that $f : A (\subseteq \mathbb{R}) \to \mathbb{R}$ is uniformly continuous if and only if for all sequences $(x_n) \subseteq A$, $(y_n) \subseteq A$, $|x_n - y_n| \to 0$ implies $|f(x_n) - f(y_n)| \to 0$. 5

UNIT—V

24. (a) State and prove Taylor's theorem with Cauchy's form of remainder. 1+4=5

(b) Find the Maclaurin's series expansion for $\cos x$ and show that it converges to $\cos x$. 5

25. (a) Let I be an interval, let x_0 be an interior point of I, and let $n \geq 2$. Suppose that the derivatives $f', f'', \cdots, f^{(n)}$ exist and are continuous in a neighbourhood of x_0 and that $f'(x_0) = \cdots = f^{(n-1)}(x_0)$, but $f^{(n)}(x_0) \neq 0$. Show that—

(i) if n is even and $f^{(n)}(x_0) > 0$, then f has a relative minimum at x_0;

(ii) if n is even and $f^{(n)}(x_0) < 0$, then f has a relative maximum at x_0;

(iii) if n is odd, then f does not have relative maximum or relative minimum. 6

(b) Show that for $x > 0$

$$1 + \frac{1}{2}x - \frac{1}{8}x^3 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$$ 4

★★★