2019/TDC/EVEN/MTMHC-201T/029

TDC (CBCS) Even Semester Exam., 2019

MATHEMATICS

(2nd Semester)

Course No. : MTMHCC-201T

(Real Analysis)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* questions : 1×2=2

(a) Find the supremum of the set

$$\left\{1+\frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$$

(b) State completeness property of $\mathbb{R}$.

(c) Let

$$I_n = \left(0, \frac{1}{n}\right)$$

for $n \in \mathbb{N}$. Find the value of

$$\bigcap_{n=1}^{\infty} I_n$$

Either

2. (a) Prove that the following statements are equivalent : 6

(i) S is countable set.

(ii) There exists a surjection of $\mathbb{N}$ onto S.

(iii) There exists an injection of S into $\mathbb{N}$.

(b) Prove that

(i) If $a \in \mathbb{R}$, $a \neq 0$, then $a^2 > 0$

(ii) $1 > 0$

(iii) If $n \in \mathbb{N}$, then $n > 0$ 6

Or

3. (a) State and prove Archimedean property. Also show that if $t > 0$, there exists $n_t \in \mathbb{N}$ such that

$$0 < \frac{1}{n_t} < t$$

1+3+2=6

(b) If x and y are any real numbers with $x < y$, then prove that there exists a rational number $r \in Q$ such that $x < r < y$. 4

(c) If

$$S = \left\{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\right\}$$

find inf S and sup S. 2

UNIT—II

4. Answer any *two* questions : 1×2=2

(a) Give an example of an open set which is not an interval.

(b) Find the derived set of the set

$$A = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$$

(c) Give an example of a set which is neither open nor closed.

Either

5. (a) State and prove Bolzano-Weierstrass theorem for sets. 1+5=6

(b) Define limit point of a set. Obtain the derived set of

$$\left\{\frac{1}{m}+\frac{1}{n} : m \in \mathbb{N},\ n \in \mathbb{N}\right\}$$ 1+1=2

(c) Prove that the derived set of a set is closed. 4

Or

6. (a) Prove that intersection of any finite number of open sets is open.

The above statement may not be true for arbitrary family of open sets. Justify with counterexample. 4+2=6

(b) Define closure of a set. Prove that the closure of a set S is the intersection of all closed supersets of S. 1+5=6

UNIT—III

7. Answer any *two* questions : 1×2=2

(a) Give an example of a bounded sequence that is not a Cauchy sequence.

(b) Find :

$$\lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2}$$

(c) Define monotone sequence.

Either

8. (a) Define limit of a sequence. Let $X=(x_n)$ and $Y=(y_n)$ be sequences of real numbers that converge to x and y respectively, then show that the sequence $X+Y$ and $X \cdot Y$ converges to $x+y$ and $x \cdot y$ respectively. 1+2+3=6

(b) State and prove monotone convergence theorem. 1+5=6

Or

9. (a) State and prove squeeze theorem on limits. Also prove that

$$\lim_{n\to\infty}\left(\frac{\sin n}{n}\right)=0$$ 1+3+2=6

(b) Establish the convergence or the divergence of the sequence (y_n), where

$$y_n=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$$

for $n \in \mathbb{N}$. 4

(c) Define bounded sequence. Give an example of a bounded sequence which is not convergent. 1+1=2

UNIT—IV

10. Answer any *two* questions : 1×2=2

(a) State Bolzano-Weierstrass theorem for sequence.

(b) Give an example of an unbounded sequence that has a convergent subsequence.

(c) Define Cauchy sequence.

Either

11. (a) Let $X=(x_n)$ be a sequence of real numbers. Then prove that the following statements are equivalent : 6

(i) The sequence $X=(x_n)$ does not converge to $x \in \mathbb{R}$.

(ii) There exists an $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $|x_{n_k} - x| \geq \varepsilon_0$.

(iii) There exists an $\varepsilon_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that $|x_{n_k} - x| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.

(b) Show that the sequence $\left(\frac{1}{n}\right)$ is a Cauchy sequence. 3

(c) Show that a Cauchy sequence of real numbers is bounded. 3

Or

12. (a) Prove that a sequence of real numbers is convergent if and only if it is a Cauchy sequence. 6

(b) Show that the sequence (y_n), where

$$y_n = \frac{1}{\lfloor 1} - \frac{1}{\lfloor 2} + \cdots + \frac{(-1)^{n+1}}{\lfloor n}$$

is convergent. 3

(c) Show that a bounded, monotone increasing sequence is a Cauchy sequence. 3

UNIT—V

13. Answer any *two* questions : 1×2=2

(a) State the necessary condition for convergence of an infinite series Σu_n.

(b) If Σa_n with $a_n > 0$ is convergent, then is Σa_n^2 always convergent?

(c) Give an example of a series which is convergent but not absolutely convergent.

Either

14. (a) If Σu_n is a positive term series such that

$$\lim_{n \to \infty} (u_n)^{\frac{1}{n}} = l$$

then prove that the series

(i) is convergent if $l < 1$

(ii) is divergent, if $l > 1$

(iii) the test fails to give any definite information, if $l = 1$ 6

(b) Test the behaviour of the following series (any *two*) : 3×2=6

(i) $\frac{\alpha}{\beta} + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + \cdots$, $\alpha, \beta \in \mathbb{R}$

(ii) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$

(iii) $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots$

Or

15. *(a)* State and prove Leibnitz test. 1+5=6

(b) Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$$

is conditionally convergent. 3

(c) Prove that every absolutely convergent series is convergent. 3

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