TDC (CBCS) Odd Semester Exam., 2019

MATHEMATICS

(1st Semester)

Course No. : MTMHCC–102T

(Higher Algebra)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* from the following questions : 2×2=4

(a) Expand $\cos^2\theta$ in the powers of θ.

(b) If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$, then prove that

$$x_1 x_2 \cdots \text{ to infinity} = -1$$

(c) Show that

$$\pi = 2\sqrt{3}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5}\cdot\frac{1}{3^2} - \frac{1}{7}\cdot\frac{1}{3^3} + \cdots\right)$$

2. Answer *either* [*(a)* and *(b)*] *or* [*(c)* and *(d)*] :

(a) If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and $x + y + z = xyz$, then prove that

$$\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) + 1 = 0$$ 5

(b) *(i)* Show that the product of n, nth roots of unity is $(-1)^{n-1}$. 3

(ii) Show that

$$\frac{\pi}{12} = \left(1 - \frac{1}{3^{1/2}}\right) - \frac{1}{3}\left(1 - \frac{1}{3^{3/2}}\right) + \frac{1}{5}\left(1 - \frac{1}{3^{5/2}}\right) - \cdots \infty$$ 2

(c) Prove that the principal value of $(\alpha + i\beta)^{x+iy}$ is wholly real or wholly imaginary according as $\frac{1}{2}y\log(\alpha^2 + \beta^2) + x\tan^{-1}\frac{\beta}{\alpha}$ is an even or odd multiple of $\frac{\pi}{2}$. 5

(d) If x_1, x_2, x_3, x_4 are the roots of the equation

$$x^4 - x^3\sin 2\alpha + x^2\cos 2\alpha - x\cos\alpha - \sin\alpha = 0$$

then show that

$$\sum \tan^{-1} x_1 = n\pi + \frac{\pi}{2} - \alpha$$ 5

UNIT—II

3. Answer any *two* from the following questions : 2×2=4

(a) Prove that inverse of function if it exists is unique.

(b) Give an example of a relation which is symmetric and transitive but not reflexive.

(c) Define the following :

(i) Equivalence class

(ii) Portion of a set

4. Answer *either* [*(a)* and *(b)*] *or* [*(c)* and *(d)*] :

(a) Define bijective mapping. Show that $f : N \to N$ defined by

$$f(x) = \begin{cases} x + 1, & \text{if } x \text{ is odd} \\ x - 1, & \text{if } x \text{ is even} \end{cases}$$

is bijective. 5

(b) Let $f : Z \to Z$ be defined by $f(n) = 3n$ for all $n \in Z$ and $g : Z \to Z$ be defined by

$$g(n) = \begin{cases} \frac{n}{3}, & \text{if } n \text{ is a multiple of 3} \\ 0, & \text{if } n \text{ is not a multiple of 3} \end{cases}$$

$\forall n \in Z$. Show that

$$g \circ f = I_Z \text{ and } f \circ g \neq I_Z$$ 5

(c) Consider $f : R \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$, where R^+ is the set of all non-negative real numbers. Show that f is invertible with

$$f^{-1}(y) = \frac{\sqrt{y+6} - 1}{3}$$ 4

(d) Let $f : X \to Y$ be a map such that $A, B \subseteq X$. Then show that

(i) $f(A \cup B) = f(A) \cup f(B)$

(ii) $f(A \cap B) \subseteq f(A) \cap f(B)$

The equality hold provided f is 1–1 map.

3+3=6

UNIT—III

5. Answer any *two* from the following questions : 2×2=4

(a) If $a \equiv b (\text{mod}\, m)$, then prove that

$$a^K \equiv b^K (\text{mod}\, m) \ \forall\ K \geq 1$$

(b) Find the remainder when 5^{48} is divided by 24.

(c) Prove that one of every three consecutive integers is divisible by 3.

6. Answer *either* [*(a)* and *(b)*] *or* [*(c)* and *(d)*] :

(a) State and prove division algorithm. 1+4=5

(b) If a and b are any two integers not both zero, then $\gcd(a, b)$ exists and is unique. 5

(c) Prove that well-ordering principle is equivalent to the principle finite induction. 4

(d) *(i)* If a, b and c are integers such that $ac \equiv bc (\text{mod}\, m)$, $m > 0$ is a fixed integer and $d = (c, m)$, then show that $a \equiv b\left(\text{mod}\,\frac{m}{d}\right)$. 3

(ii) Prove that the product of any three consecutive integers is divisible by $\underline{|3}$. 3

UNIT—IV

7. Answer any *two* from the following questions : 2×2=4

(a) Find the equation whose roots are double the roots of

$$x^3 - 6x^2 + 11x - 6 = 0$$

(b) Remove the second term of the equation

$$x^4 + 10x^3 + 26x^2 + 10x + 1 = 0$$

(c) State Descartes' rule of signs.

8. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) If one root of the equation $x^3 + px^2 + qx + r = 0$ equals the sum of the other two, then prove that

$$p^3 + 8r = 4pq$$ 5

(b) Solve by Cardan's method :

$$x^3 - 18x - 35 = 0$$ 5

(c) If α, β, γ be the roots of the equation $x^3 + 2x^2 + 3x + 4 = 0$, then find the equation whose roots are

$$\alpha - \frac{1}{\beta\gamma},\ \beta - \frac{1}{\gamma\alpha},\ \gamma - \frac{1}{\alpha\beta}$$ 4

(d) If $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n x^n + p_n = 0$$

$p_n \neq 0$, then find the value of

(i) $\sum \frac{\alpha_1^2 + \alpha_2^2}{\alpha_1 \alpha_2}$

(ii) $\sum \frac{\alpha^1}{\alpha_2^2}$

3+3=6

UNIT—V

9. Answer any *two* from the following questions : 2×2=4

(a) Prove that any subset of LI set of vectors is LI.

(b) Define linearly independent and linearly dependent set of vectors.

(c) Define echelon form of a matrix.

10. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Prove that the rank of product of two matrices cannot exceed the rank of either matrix. 5

(b) Find the rank of the matrix

$$\begin{pmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$$ 5

(c) Solve by Gaussian elimination method :

$$x + y + z = 6$$
$$x - y + z = 2$$
$$2x + y - z = 1$$ 5

(d) (i) Show that the vectors (1, 2, −3, 4), (3, −1, 2, 1) and (1, −5, 8, −7) of $R^4(R)$ are linearly dependent. 3

(ii) Show that the vectors (2, 3, 4), (0, 5, 6) and (0, 0, 8) are linearly independent. 2

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