2022/TDC/ODD/SEM/MTMDSE-501T (A/B/C/D)/331

TDC (CBCS) Odd Semester Exam., 2022

MATHEMATICS

(5th Semester)

Course No. : MTMDSE-501T

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

Candidates have to answer *either* from Option—A *or* Option—B *or* Option—C *or* Option—D

OPTION—A

(For Honours Students only)

Course No. : MTMDSE-501T (A)

(Mechanics)

UNIT—I

1. Answer any *four* of the following as directed : 1×4=4

   (a) If P and Q are parallel forces, then their resultant is a _____ force. (Fill in the blank)

   (b) When are three coplanar forces said to be in equilibrium?

   (c) Define force of friction.

(d) Define coefficient of friction.

(e) Define angle of friction.

2. Answer any *one* of the following : 2

(a) What do you mean by limiting equilibrium and limiting friction?

(b) Write law of dynamical friction.

Answer any *one* of the following : 8

3. (a) A heavy uniform rod of length a rests with one end against a smooth vertical wall, the other end being tied to a point of the wall by a string of length l. Prove that the rod may remain in equilibrium at an angle θ to the wall, given by

$$\cos^2\theta = \frac{l^2 - a^2}{3a^2}$$ 3

(b) Equal weights P and P are attached to two strings ACP and BCP passing over a smooth peg C. $\overline{AB}$ is a heavy beam of weight W, whose centre of gravity is at distance a meter from A and b meter from B. Show that $\overline{AB}$ is inclined to the horizon at an angle

$$\tan^{-1}\left[\frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right)\right]$$ 5

4. (a) A uniform ladder is in equilibrium with one end resting on the ground and the other against a vertical wall. If the ground and wall be both rough, the coefficient of friction being μ and μ' respectively and if the ladder be on the point of slipping at both ends, then show that the inclination of the ladder to the horizon is given by

$$\tan\theta = \frac{1-\mu\mu'}{2\mu}$$ 4

(b) Two weights P and Q $(P > Q)$ are placed on a rough inclined plane, being connected by a thin string passing over a small smooth pulley on plane, the parts of the string being parallel to the line of greatest slope. The inclination of the plane to the horizon is gradually increased. Prove that the weights will being to slip on the planes when its inclination θ to the horizon is given by

$$\tan\theta = \frac{P+Q}{P-Q}\tan\lambda$$

where λ is the angle of friction of the plane assumed same with respect to either weight. 4

UNIT—II

5. Answer any *four* of the following : 1×4=4

(a) What will be radial and transverse component of velocity if the motion of a particle in a circular path whose centre is at origin and radius is a?

(b) Write the expression for tangential and normal component of acceleration.

(c) What will be tangential acceleration of a particle if it is moving with uniform velocity?

(d) Define amplitude of SHM.

(e) Write the expression for periodic time of SHM.

6. Answer any *one* of the following : 2

(a) A particle describing SHM in a straight line has velocities v_1, v_2, v_3 at distances x_1, x_2, x_3 from the centre of the path. Prove that

$$x_1^2(v_2^2 - v_3^2) + x_2^2(v_3^2 - v_1^2) + x_3^2(v_1^2 - v_2^2) = 0$$

(b) If the velocity of a point moving in a plane curve varies as the radius of curvature, show that the direction of motion revolves with constant angular velocity.

Answer any *one* of the following : 8

7. *(a)* Obtain the expressions for radial and transverse component of velocity of a particle moving in a plane curve. 5

(b) Prove that if the tangential and normal accelerations of a particle describing a plane curve be constant throughout the motion, the angle ψ which the direction of motion turns in time t is given by $\psi = A\log(1 + Bt)$. 3

8. *(a)* A particle oscillates with SHM of amplitude a and periodic time T. Find the expression for the velocity v *(i)* in terms of a, T and x, *(ii)* in terms of a, T, t and also prove that

$$\int_0^T v^2 dt = \frac{2\pi^2 a^2}{T}$$ 5

(b) A point moves on a parabola

$$\frac{2a}{r} = 1 + \cos\theta$$

in such a manner that the component of velocity at right angle to the radius vector from the focus is constant. Show that the acceleration of the point is constant in magnitude. 3

UNIT—III

9. Answer any *four* of the following as directed : 1×4=4

(a) Write the formula of inverse square law.

(b) Each planet describes an _____ with the sun as one of its _____.

(Fill in the blanks)

(c) Define areal velocity.

(d) If m_1 and m_2 are masses of two particles distant r apart, then write the force between them.

(e) A particle moves in a path so that its acceleration is always directed to a fixed point and is equal to

$$\frac{\mu}{(\text{distance})^2}$$

Then the path is a _____ _____.

(Fill in the blanks)

10. Answer any *one* of the following : 2

(a) A particle describes an ellipse under a force

$$\frac{\mu}{(\text{distance})^2}$$

towards the focus. If it was projected with velocity v from a point distance r from the centre of force, show that its periodic time is

$$\frac{2\pi}{\sqrt{\mu}}\left[\frac{2}{r}-\frac{v^2}{\mu}\right]^{-\frac{3}{2}}$$

(b) A particle moves towards a centre of attraction starting from rest at a distance a from the centre, if its velocity when at any distance x from the centre varies as

$$\sqrt{\frac{a^2-x^2}{x^2}}$$

Find the law of force.

Answer any *one* of the following : 8

11. (a) Prove that time taken by the earth to travel over half of its orbit separated by the minor axis remote from the sun is two days more than half a year. Given the period of the earth is 365 days and the eccentricity of the orbit = $1/60$. 4

(b) If a planet were suddenly stopped in its orbit supposed circular, show that it would fall into the sun in a time which is $\frac{\sqrt{2}}{8}$ times the period of the planet's revolution. 4

12. *(a)* A particle is attracted by a force to a fixed point varying inversely as the $(\text{distance})^{4/3}$. Show that the velocity acquired in falling from rest from an infinite distance to a distance a from the centre is equal to the velocity acquired in falling from rest at a distance a to a distance $a/8$ from the centre. 4

(b) If h be the height due to a velocity v at the earth's surface supposing its attraction constant, and H the corresponding height when the variation of gravity is taken into account, prove that $\frac{1}{h}-\frac{1}{H}=\frac{1}{r}$, where r is radius of the earth. 4

UNIT—IV

13. Answer any *four* of the following : 1×4=4

(a) Define work.

(b) Define power.

(c) Define impulse of a force.

(d) What do you mean by direct impact?

(e) Write the values of coefficient of elasticity if the spheres be perfectly *(i)* elastic and *(ii)* inelastic.

14. Answer any *one* of the following : 2

(a) A sphere impinges directly on an equal sphere at rest. If the coefficient of restitution be e, then prove that their velocities after impact are in the ratio $(1-e):(1+e)$.

(b) State principle of conservation of linear momentum.

Answer any *one* of the following : 8

15. *(a)* A shell of mass M is moving with velocity v. An internal explosion generates an amount of energy E and breaks the shell into two portions whose masses are in the ratio $m_1 : m_2$. The fragments continue to move in the original line of motion of the shell. Prove that their velocities are

$$v+\sqrt{\frac{2m_2E}{m_1M}} \text{ and } v-\sqrt{\frac{2m_1E}{m_2M}}$$ 4

(b) A body of mass 10 kg and velocity 9 m/sec strikes another body of mass 20 kg and velocity 6 m/sec moving in the same direction. If the velocity of the first body becomes 6 m/sec after impact, find the coefficient of restitution and loss of kinetic energy. 4

16. *(a)* Find loss of kinetic energy by direct impact of two smooth spheres. 5

(b) A gun of mass M fires a shell of mass m horizontally and the energy of explosion is such as would be sufficient to project the shell vertically to a height h. Prove that the velocity of recoil is

$$\left\{\frac{2m^2gh}{M(M+m)}\right\}^{\frac{1}{2}}$$

3

UNIT—V

17. Answer any *four* of the following : 1×4=4

(a) Define moment of inertia of a system of particle.

(b) Define product of inertia.

(c) What is the MI of a circular ring of radius a and mass M about a diameter?

(d) What do you mean by effective force?

(e) Write MI of a rectangular plate of sides $2a$, $2b$ and mass M about a line through the centre and parallel to the side $2a$.

18. Answer any *one* of the following : 2

(a) If A, B, C denote the moments of inertia of the body about three mutually perpendicular axes, then show that $A > 2D$, $B > 2E$, $C > 2F$.

(b) Find MI of a uniform rod of length $2a$.

Answer any *one* of the following : 8

19. *(a)* Find MI of a body about a line. 5

(b) Find MI of a rectangular parallelopiped about an edge. 3

20. State and prove parallel axes theorem. 8

OPTION—B

(For Honours Students only)

Course No. : MTMDSE-501T (B)

(Number Theory)

UNIT—I

1. Answer any *four* of the following : 1×4=4

(a) Does the equation $2x+10y=17$ have any solution in the positive integers? Justify.

(b) State the fundamental theorem of arithmetic.

(c) Under what condition on c does the congruence $ac \equiv bc \pmod{n}$ implies $a \equiv b \pmod{n}$?

(d) If p is a prime, what is the remainder when $(p-1)!$ is divided by p?

(e) What is the units digit in 3^{1000}?

2. Answer any *one* of the following : 2

(a) Find the remainder when $1!+2!+\cdots+50!$ is divided by 12.

(b) Solve the congruence $x \equiv 2 \pmod{3}$, $x \equiv 3 \pmod{5}$ by Chinese remainder theorem.

3. Answer *either (a)* and *(b) or (c)* and *(d)* :

(a) Find the positive solutions of the linear Diophantine equation $18x+5y=48$. 4

(b) If p_n is the nth prime, then show that $p_n \le 2^{2^{n-1}}$. 4

(c) If $\gcd(a, n)=1$, show that the integers

$$c, c+a, c+2a, \cdots, c+(n-1)a$$

form a complete set of residues modulo n for any c. 4

(d) State and prove Fermat's little theorem. 1+3=4

UNIT—II

4. Answer any *four* of the following : 1×4=4

(a) Let $\tau(n)$ denotes the number of positive divisors of n. Find $\tau(180)$.

(b) State Möbius inversion formula.

(c) Find $\sum_{d|180} d$.

(d) Let μ be the Möbius function. For $n > 1$, what is the value of $\sum_{d|n} \mu(d)$?

(e) Write the formula of $\sigma(n)$ in terms of the prime divisors of n.

5. Answer any *one* of the following : 2

(a) Show that

$$\prod_{d|n} d = n^{\frac{1}{2}\tau(n)}$$

(b) Show that the Möbius function μ is multiplicative.

6. Answer *either (a)* and *(b) or (c)* and *(d)* : 8

(a) Let F and f be two number theoretic functions related by

$$F(n) = \sum_{d|n} f(d)$$

Then show that

$$f(n) = \sum_{d|n} \mu(d)\, F\left(\frac{n}{d}\right)$$ 4

(b) Show that

$$\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n}$$ 4

(c) Show that the number of positive divisor function τ is multiplicative. 4

(d) If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of n, then show that

$$1 > \frac{\sigma(n)}{n} > \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_r}\right)$$ 4

UNIT—III

7. Answer any *four* of the following : 1×4=4

(a) Find $[-\pi]$, where $[\cdot]$ is the greatest integer function.

(b) What is $\varphi(125)$, where φ is the Euler's function?

(c) Find the last digit in 7^{234}.

(d) Evaluate

$$\sum_{d|20} \varphi(d)$$

(e) Write down a reduced set of residues modulo 20.

8. Answer any *one* of the following : 2

(a) Find the highest power of 5 that divides 1000!

(b) Show that $\varphi(n)$ is even for $n > 2$.

9. Answer *either* (a) and (b) *or* (c) and (d) : 8

(a) Show that product of r consecutive positive integers is divisible by $r!$ 4

(b) If $\gcd(a, n) = 1$, show that $a^{\varphi(n)} \equiv 1 \pmod{n}$. 4

(c) For $n > 1$, show that the sum of positive integers less than n and relatively prime to n is $\frac{1}{2}n\varphi(n)$. 4

(d) Find the remainder when 2^{100000} is divided by 77. 4

UNIT—IV

10. Answer any *four* of the following : 1×4=4

(a) Find the order of 2 modulo 7.

(b) Evaluate the Legendre symbol

$$\left(-\frac{46}{17}\right)$$

(c) State quadratic reciprocity law.

(d) Give example of a composite number having no primitive root.

(e) Give an example of a composite number having at least one primitive root.

11. Answer any *one* of the following : 2

(a) Show that

$$\left(-\frac{1}{p}\right) = (-1)^{\frac{1}{2}(p-1)}$$

for any prime p.

(b) List all positive integers which have at least one primitive root.

12. Answer *either* (a) and (b) *or* (c) and (d) : 8

(a) If $\gcd(m, n) = 1$, $m, n > 2$, show that mn has no primitive roots. 4

(b) Show that there are infinitely many primes of the form $4k + 1$, $k \geq 1$. 4

(c) State Gauss lemma. Using it or otherwise, show that

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$$ 4

(d) Prove quadratic reciprocity law. 4

UNIT—V

13. Answer any *four* of the following : 1×4=4

(a) Which numbers are called Mersenne numbers?

(b) What is the value of $\sigma(n)$ for perfect numbers?

(c) What can be the last digit in a perfect number?

(d) State Fermat's last theorem.

(e) Give example of an amicable pair of numbers.

14. Answer any *one* of the following : 2

(a) Write the general solutions of the equation $x^2 + y^2 = z^2$.

(b) If $2^k - 1$ is a prime number for some $k > 1$, show that $2^{k-1}(2^k - 1)$ is a perfect number.

15. Answer *either (a)* and *(b) or (c)* and *(d)* : 8

(a) Show that every even perfect number must be of the form $2^{k-1}(2^k - 1)$, where $2^k - 1$ is prime and $k > 1$. 4

(b) Is every Fermat number necessarily prime? Justify your answer. 4

(c) Show that every even perfect number either ends with 6 or 8. 4

(d) For $n \geq 1$, if the Fermat number $F_n = 2^{2^n} + 1$ is a prime number, then show that

$$3^{\frac{F_n - 1}{2}} \equiv -1 \pmod{F_n}$$ 4

OPTION—C

(For Pass Students only)

Course No. : MTMDSE-501T (C)

(Matrices)

UNIT—I

1. Answer any *four* of the following as directed : 1×4=4

(a) Define subspace of a vector space.

(b) A set containing linearly independent set of vectors is itself linearly independent.

(Write True or False)

(c) What do you mean by linear combination of vectors?

(d) What is basis?

(e) Is the set of vectors

$$V = \{(a, b, c) \in \mathbb{R}^3 \mid a = b = c\}$$

a subspace of $\mathbb{R}^3$ over $\mathbb{R}$?

2. Answer any *one* of the following : 2

(a) Show that the vectors (1, 2, 4), (3, 6, 12) are linearly dependent.

(b) Express each of the standard basis vectors of R^3 as a linear combination of $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$, $\alpha_3 = (0, -3, 2)$.

Answer any *one* of the following : 8

3. (a) Show that the intersection of an arbitrary collection of subspaces of a vector space V is also a subspace of V. 5

(b) Express $v = (1, -2, 5)$ in $\mathbf{R}^3$ as a linear combination of the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$, $u_3 = (2, -1, 1)$. 3

4. (a) Prove that the necessary and sufficient condition for a non-empty subset W of a vector space $V(F)$ to be a subspace of V is $\alpha x + \beta y \in W \ \forall \ \alpha, \beta \in F$ and $x, y \in W$. 5

(b) Show that the vectors $(1, 2, 1)$, $(2, 1, 0)$, $(1, -1, 2)$ form a basis of R^3. 3

UNIT—II

5. Answer any *four* of the following : 1×4=4

(a) What do you mean by invariant subspace of a vector space?

(b) If λ is an eigenvalue of a matrix A and matrix B is similar to A, then is λ an eigenvalue of B?,

(c) Define eigenvector of a linear operator.

(d) What is the reflection of the point (3,2) about X-axis?

(e) Give one eigenvalue of a singular matrix.

6. Answer any *one* of the following : 2

(a) What do you mean by eigenvalue and eigenspace of a matrix?

(b) If T is any linear operator on a vector space, then show that the null space of T is invariant under T.

Answer any *one* of the following : 8

7. (a) Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$ 4

(b) Show that the matrices A and A^T have the same eigenvalues. 4

8. (a) Prove that if the characteristic roots of A are $\lambda_1, \lambda_2, \cdots, \lambda_n$, then the characteristic roots of A^2 are $\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2$. 4

(b) Show that the space generated by $(1, 1, 1)$ and $(1, 2, 1)$ is invariant subspace of $\mathbb{R}^3$ under T, where

$$T(x, y, z) = (x + y - z, \ x + y, \ x + y - z)$$ 4

UNIT—III

9. Answer any *four* of the following as directed : 1×4=4

(a) Define unitary matrix.

(b) Find the rank of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(c) Define skew-Hermitian matrix.

(d) Define rank of matrix.

(e) If A^{θ} and B^{θ} be the transposed conjugate of matrices A and B respectively, then $(AB)^{\theta} =$ ______.

(Fill in the blank)

10. Answer any *one* of the following : 2

(a) If the non-singular matrix A is symmetric, then prove that A^{-1} is also symmetric.

(b) If the points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear, then show that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} < 3$$

Answer any *one* of the following : 8

11. (a) Reduce the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

to normal form. 4

(b) Prove that the rank of the transpose of a matrix is the same as that of the original matrix. 4

12. (a) If A and B are symmetric matrices, then show that AB is symmetric iff $AB = BA$. 4

(b) If A be any square matrix, then show that $(A + A^T)$ is symmetric and $A - A^T$ is skew-symmetric. 4

UNIT—IV

13. Answer any *four* of the following as directed : 1×4=4

(a) What is a diagonal matrix?

(b) Write the necessary and sufficient condition for a matrix to be invertible.

(c) Every invertible matrix possesses a unique inverse.

(Write True or False)

(d) Is the matrix

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

diagonalizable?

(e) Find the inverse of

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

14. Answer any *one* of the following : 2

(a) If A be an $n \times n$ non-singular matrix, then show that $(A^T)^{-1} = (A^{-1})^T$.

(b) Show that the inverse of a matrix is unique.

Answer any *one* of the following : 8

15. (a) If A, B are non-singular matrices of same order, then show that $(AB)^{-1} = B^{-1}A^{-1}$. 4

(b) Show that the necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$. 4

16. (a) Compute the inverse of the following matrix by using elementary row operations : 4

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

(b) Transform the matrix

$$\begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

into diagonal form. 4

UNIT—V

17. Answer any *four* of the following as directed : 1×4=4

(a) What is the rank of a non-singular matrix of order n?

(b) When a system of linear equation has infinite number of solutions?

(c) What is augmented matrix?

(d) If A be an n-rowed non-singular matrix, X be an $n \times 1$ matrix, B be an $n \times 1$ matrix, then the system of equations $AX = B$ has infinite solutions.

(Write True or False)

(e) Under what condition a system of equation $AX = B$ possesses a solution?

18. Answer any *one* of the following : 2

(a) Show that the equation

$$2x+6y+11=0$$
$$6x+20y-6z+3=0$$
$$6y-18z+1=0$$

are not consistent.

(b) When a system of equation $AX=B$ has (i) no solution and (ii) unique solution?

Answer any *one* of the following : 8

19. (a) Show that the equations

$$x+y+z=6$$
$$x+2y+3z=14$$
$$x+4y+7z=30$$

are consistent and solve them. 4

(b) Determine the rank of the following matrix : 4

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

20. For what values of λ, the equations

$$x+y+z=1$$
$$x+2y+4z=\lambda$$
$$x+4y+10z=\lambda^2$$

have a solution and solve them in each case? 8

OPTION—D

(For Pass Students only)

Course No. : MTMDSE–501T (D)

(Linear Algebra)

UNIT—I

1. Answer any *four* of the following as directed : 1×4=4

(a) Write down the necessary and sufficient condition for a non-empty subset *W* of a vector space *V*(*F*) to be a subspace of *V*.

(b) Prove that, in a vector space, a set containing zero vector is always linearly dependent.

(c) Define dimension of a vector space.

(d) Write an example of basis of the vector space $\mathbb{R}^2(\mathbb{R})$.

(e) Superset of a linearly dependent set is linearly independent.

(Write True or False)

2. (a) Prove that intersection of two subspaces of a vector space is also a subspace. 2

Or

(b) Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

3. Answer *either* (a) and (b) *or* (c) and (d) : 8

(a) If W is a subspace of a finite dimensional vector space $V(F)$, then prove that $\dim \frac{V}{W} = \dim V - \dim W$. 5

(b) Show that the three vectors $(1, 1, -1)$, $(2, -3, 5)$ and $(-2, 1, 4)$ of R^3 are linearly independent. 3

(c) Prove that the union of two subspaces of a vector space is also a subspace if and only if one of them contains the other. 4

(d) Prove that every finitely generated vector space has a basis. 4

UNIT—II

4. Answer any *four* of the following as directed : 1×4=4

(a) Define linear transformation.

(b) Let $T: R^4 \to R^3$ be a linear transformation such that rank of T is 2. What is the nullity of T?

(c) Dimension of range space is called ____.

(Fill in the blank)

(d) Define nullity of a linear transformation.

(e) Let $T: U \to V$ be a linear transformation from a finite dimensional vector space U into another vector V, then write down the relation between rank(T) and nullity(T).

5. (a) Show that $T: R^3(R) \to R^2(R)$ defined by $T(x, y, z) = (y, z)$ is a linear transformation. 2

Or

(b) The mapping $T: R^2 \to R^3$ defined as $T(a, b) = (a - b, b - a, -a)$ is a linear transformation from R^2 into R^3. Find the nullity of T.

6. Answer *either* (a) and (b) *or* (c) and (d) : 8

(a) State and prove Sylvester's law of nullity. 1+4=5

(b) Let $T: V(F) \to U(F)$ is a linear transformation, then show that T is one-one if and only if $\ker T = \{0\}$. 3

(c) Let $V(F)$ be a vector space and $T: V \to V$ is a linear operator, then prove that the following are equivalent : 4

(i) $\text{Range}(T) \cap \ker(T) = \{0\}$

(ii) $T(T_x) = 0 \Rightarrow T_x = 0$

(d) Prove that there exists a linear transformation $T: R^2 \to R^3$ such that $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$. 4

UNIT—III

7. Answer any *four* of the following as directed : 1×4=4

(a) Define isomorphism on a vector space.

(b) Give an example of isomorphism on a vector space.

(c) Two finite dimensional vector spaces over the same field are isomorphic if and only if they are of the same dimension.

(Write True or False)

(d) Define invertible linear transformation.

(e) Define singular linear transformation.

8. (a) Let T be a linear transformation from a vector space $U(F)$ into a vector space $V(F)$, then show that T is one-one if T is non-singular. 2

Or

(b) Let T be an invertible linear transformation on a vector space $V(F)$, then show that $T^{-1}T = I$.

9. Answer *either* (a) and (b) *or* (c) and (d) : 8

(a) Let U and V be vector spaces over the field F and let T be a linear transformation from U into V. If T is one-one and onto, then show that T^{-1} is a linear transformation from V into U. 3

(b) Prove that every vector space over F of dimension n is isomorphic to the vector space $F^n(F)$, where F is a field. 5

(c) Let T be a linear transformation from a vector space $U(F)$ into a vector space $V(F)$. If S is a subspace of U, then prove that $T(S)$ is a subspace of V. 3

(d) Prove that a linear transformation from a finite dimensional vector space to another finite dimensional vector space of same dimension is one-one if and only if it is onto. 5

UNIT—IV

10. Answer any *four* of the following as directed : 1×4=4

(a) A square matrix A and its transpose A^t have the same characteristic polynomial.

(Write True or False)

(b) If v is an eigenvector of a linear operator T corresponding to eigenvalue λ, then show that αv is also an eigenvector for any scalar $\alpha \neq 0$.

(c) What are the eigenvalues of a triangular matrix?

(d) Find the characteristic roots of the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(e) Show that 0 is an eigenvalue of a linear operator T on a vector space $V(F)$ if T is singular.

11. (a) Let $V(F)$ be a vector space and $T: V \to V$ is a linear operator, then show that $\lambda \in F$ is an eigenvalue of T if and only if $T - \lambda I$ is singular. 2

Or

(b) Prove that similar matrices have the same characteristic polynomial.

12. Answer *either* (a) and (b) *or* (c) and (d) : 8

(a) State and prove Cayley-Hamilton theorem. 1+5=6

(b) Show that eigenvalues of a unitary matrix are of unit modulus. 2

(c) Find the characteristic equation of the matrix

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 4 & -1 \end{pmatrix}$$

and find A^{-1} with the help of Cayley-Hamilton theorem. 4

(d) Let $V(F)$ be a vector space of dimension n and $T: V \to V$ is a linear operator, then show that T can be represented by a diagonal matrix if and only if there exists a basis of V consisting of eigenvectors of T. 4

UNIT—V

13. Answer any *four* of the following as directed : 1×4=4

(a) Let V be an inner product space, then show that $(0, x) = 0$ for all x in V.

(b) Write down the triangle in equality in inner product space V.

(c) If x and y are vectors in an inner product space, then write down the value of $||x+y||^2 + ||x-y||^2$.

(d) In an inner product space, if $||x+y|| = ||x|| + ||y||$, then the vectors x and y are linearly independent/linearly dependent.

(Write the correct answer)

(e) If x is orthogonal to y, then show that αx is also orthogonal to y, where α is a scalar.

14. *(a)* Let V be an inner product space and x, y be vectors in V, then show that $x = y$ if and only if $(x, z) = (y, z)$ for every z in V. 2

Or

(b) Define orthogonality of two vectors. Also define orthogonal set. 1+1=2

15. Answer *either (a)* and *(b) or (c)* and *(d)* : 8

(a) In an inner product space $V(F)$, prove that $|(x, y)| \leq ||x||\,||y||$. 5

(b) If $\{v_1, v_2, \cdots, v_n\}$ is an orthonormal subset of an inner product space V, then prove that for any $v \in V$, the vector

$$v - \sum_{i=1}^{n} (v, v_i)\, v_i$$

is perpendicular to each v_i. 3

(c) State and prove Bessel's inequality in an inner product space. 1+5=6

(d) In an inner product space $V(F)$, prove that $||\alpha|| \geq 0$ and $||\alpha|| = 0$ if and only if $\alpha = 0$. 2

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