2020/TDC(CBCS)/ODD/SEM/
MTMDSE–501T (A/B/C/D)/333B

TDC (CBCS) Odd Semester Exam., 2020
held in March, 2021

MATHEMATICS

(5th Semester)

Course No. : MTMDSE–501T

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

Honours students will answer either Group—A or Group—B and Pass students will answer either Group—C or Group—D

GROUP—A

(For Honours students)

Course No. : MTMDSE–501T (H)

(NUMBER THEORY)

SECTION—A

Answer any *twenty* of the following questions : 1×20=20

1. Does the linear diophantine equation $12x + 21y = 5$ have a solution? Justify.

2. What is the remainder when 2^{2021} is divided by 31?

3. What is the Goldbach's conjecture?

4. State Fermat's little theorem.

5. What is the remainder when 96! is divided by 97?

6. If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, show that $a \equiv c \pmod{n}$.

7. Solve the linear congruence $3x \equiv 7 \pmod{10}$.

8. What is the largest power of 10 that divides 100!?

9. Evaluate $\tau(2020)$.

10. Evaluate $\sigma(216)$.

11. How many positive divisors does 97^{365} have?

12. What is a multiplicative function?

13. When $n = 206$, show that $\sigma(n) = \sigma(n+1)$.

14. What is the smallest positive integer having exactly 6 positive divisors?

15. Check if there is any positive integer n for which $\sigma(n) = 5$.

16. If n is a perfect square, show that $\tau(n)$ is odd.

17. Does there exist a positive integer n such that $\phi(n) = 1729$?

18. Evaluate $\phi(1001)$.

19. Using Euler's theorem, find the unit digit of 7^{399}.

20. Find the highest power of 5 that divides 1000!.

21. If x is an integer, what is the value of $[x]+[-x]$?

22. If p is a prime, what is $\phi(p^2)$?

23. Verify that $\phi(n) = \phi(n+1)$, when $n = 5186$.

24. Which one of the following is greater?

$$[x]+[y] \text{ or } [x+y]$$

25. Define the order of an integer modulo n.

26. Find the order of 5 modulo 19.

27. Define primitive root of a positive integer.

28. What are the incongruent solutions of $x^2 \equiv 1 \pmod{p}$?

29. Define Legendre symbol.

30. What is the value of $(36/7)$?

31. State quadratic reciprocity law.

32. Check if 3 is a quadratic residue of 13.

33. What is a perfect number?

34. Can a perfect square be a perfect number?

35. What are Mersenne primes?

36. What are amicable numbers?

37. What are Fermat numbers?

38. State Fermat's last theorem.

39. Can the product of two odd primes be a perfect number?

40. What are the only possible unit digits of an even perfect number?

SECTION—B

Answer any *five* of the following questions : 2×5=10

41. What is the remainder when

$$1^5 + 2^5 + 3^5 + \cdots + 99^5 + 100^5$$

is divided by 4?

42. Show that there are infinitely many primes.

43. Given any $n \in \mathbb{N}$ with $n > 1$, derive a general expression to evaluate the value of $\tau(n)$.

44. Show that

$$\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n}$$

45. Use Euler's theorem to show that $a^{37} \equiv a \pmod{1729}$ for any integer a.

46. For $n > 1$, show that the sum of the positive integers less than n and relatively prime to n is $\frac{n\phi(n)}{2}$.

47. Solve the quadratic congruence

$$5x^2 - 6x + 2 \equiv 0 \pmod{13}$$

48. If $a \equiv b \pmod{p}$, then show that $(a/p) = (b/p)$.

49. If p and $q = 2p+1$ are primes, then show that either $q | M_p$ or $q | M_p + 2$, where M_p is Mersenne number.

50. Show that the Fermat number F_5 is divisible by 641.

SECTION—C

Answer *any* **five** questions

51. *(a)* Determine all solutions of the Diophantine equation $24x + 138y = 18$. 5

(b) Solve the system of linear congruences : 3

$$x \equiv 1 \pmod 3,\ x \equiv 2 \pmod 5,\ x \equiv 3 \pmod 7$$

52. *(a)* State and prove Chinese remainder theorem. 5

(b) From Fermat's theorem, deduce that for any integer $n \geq 0$

$$13 \mid 11^{12n+6} + 1$$ 3

53. *(a)* If f is a multiplicative function and F is defined by

$$F(n) = \sum_{d|n} f(d)$$

show that F is also a multiplicative function. 4

(b) Find the form of all positive integers n satisfying $\tau(n) = 10$. What is the smallest such positive integer? 4

54. *(a)* State and prove Möbius inversion formula. 5

(b) If f and g are multiplicative functions, show that the product fg is also multiplicative. 3

55. *(a)* If n is a positive integer and p is a prime, then prove that the exponent of the highest power of p that divides $n!$ is

$$\sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]$$

where $[x]$ is the greatest integer less than or equal to x. 5

(b) For $n > 2$, show that $\phi(n)$ is an even integer. 3

56. *(a)* State and prove Euler's theorem. 5

(b) If p is a prime and $k > 0$, show that

$$\phi(p^k) = p^k\left(1 - \frac{1}{p}\right)$$ 3

57. *(a)* If the integer a has order k modulo n, then prove that

$$a^i \equiv a^j \pmod n$$

iff $i \equiv j \pmod k$ 3

(b) If p is a prime and
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
where $a_n \not\equiv 0 \pmod p$ is a polynomial of degree $n \geq 1$ with integer coefficients, then show that the congruence $f(x) \equiv 0 \pmod p$ has at most n incongruent solutions modulo p. 5

58. (a) For $k \geq 3$, show that 2^k has no primitive roots. 4

(b) State and prove Euler's criterion. 4

59. (a) If p is an odd prime, then prove that any prime divisor of m_p is of the form $2kp+1$. 4

(b) For Fermat numbers F_n and F_m, where $m > n \geq 0$, show that $\gcd(F_n, F_m) = 1$. 4

60. (a) Show that the radius of the incircle of a Pythagorean triangle is an integer. 5

(b) Using Pepin's test, show that the Fermat number F_3 is prime. 3

GROUP—B

(For Honours students)

Course No. : MTMDSE-501T (H)

(MECHANICS)

SECTION—A

Answer any *twenty* of the following questions : $1 \times 20 = 20$

1. Define the resultant force.

2. Write the equation to the line of action of the resultant force.

3. What are the necessary and sufficient conditions for equilibrium of any number of coplanar forces?

4. Write the converse of Lami's theorem.

5. Give the definition of friction.

6. Define the coefficient of friction.

7. Define the angle of friction

8. Define the cone of friction.

9. What are the radial and transverse velocities?

10. Define the radial and transverse acceleration.

11. What do you mean by relative velocity of a point P with respect to another point Q?

12. Write the definition of angular velocity.

13. Write the definition of angular acceleration.

14. What is the amplitude of simple harmonic motion?

15. Define the frequency of simple harmonic motion.

16. The maximum velocity of a body moving with simple harmonic motion is 2 m/sec and its period is 1/5 sec. Find its amplitude.

17. Write the equation of motion.

18. Give the definition of motion under inverse square law.

19. Write the Newton's law of gravitation.

20. What is the necessary modification of Kepler's 3rd law?

21. What is the meaning of limiting or terminal velocity?

22. What is the cycloidal pendulum?

23. Write the equation of motion when the mass moving varies.

24. What is the equation of motion in resisting medium under gravity?

25. Write the definition of work.

26. What is the impulse of a force?

27. What is the power of the force?

28. Define the conservative force.

29. Give the definition of kinetic energy.

30. What is the principle of work-energy?

31. What is the principle of conservation of energy?

32. State the principle of conservation of linear momentum.

33. Give the definition of moment of inertia.

34. Give the definition of product of inertia.

35. What is Routh's rule for moment of inertia?

36. Write the statement of Dirichlet's theorem.

37. State parallel axis theorem.

38. State perpendicular axis theorem.

39. Define the principal axes and principal moments of inertia.

40. Write the D'Alembert's principle.

SECTION—B

Answer any *five* of the following questions : $2\times5=10$

41. If the resultant of two forces acting on a particle be at right angles to one of them, and its magnitude be $\frac{1}{3}$rd of the magnitude of the other, then show that the ratio of the larger force to the smaller force is $3:2\sqrt{2}$.

42. Write three laws of statical friction.

43. A particle is moving with a constant velocity parallel to the y-axis and a velocity proportional to y is parallel to the x-axis, then prove that it will describe a parabola.

44. Prove that the periodic time of an SHM is independent of the amplitude.

45. Show that an unresisted particle falling to the earth's surface from a great distance will acquire a velocity $\sqrt{2ga}$, where a is the radius of the earth.

46. Write the Kepler's laws of planetary motion.

47. Write the principle of linear momentum for a system of particles.

48. Write the fundamental principles of impact.

49. Show that the moment of inertia of a rod of length $2a$ about an axis perpendicular to the rod is $\frac{4}{3}Ma^2$, where M be the mass of the rod.

50. Find the moment of inertia of a rectangular lamina of sides $2a$, $2b$ and mass M about a line through the centre of the lamina and parallel to the side $2a$.

SECTION—C

Answer any *five* from the following questions : 8×5=40

51. State and prove Lami's theorem.

52. A straight uniform beam of length $2h$ rests in limiting equilibrium, in contact with a rough vertical wall of height h, with one end on a rough horizontal plane and with the other end projecting beyond the wall. If both the walls and the plane be equally rough, prove that

$$\sin 2\lambda = \sin\alpha \times \sin 2\alpha$$

where λ is the angle of friction, α is the inclination of the beam to the horizon.

53. Find the tangential and normal components of velocity and acceleration of a particle moving along a plane curve.

54. Define simple harmonic motion and obtain its equation in the form $x = a\cos(\sqrt{\mu}\, t)$. Also, obtain its periodic time.

55. If v_1 and v_2 are the linear velocities of a planet, when it is respectively nearest and farthest from the sun, then show that

$$(1-e)\, v_1 = (1+e)\, v_2$$

(*Continued*)

56. A particle of mass m is acted on by a force $m\mu\left(x + \frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance a, show that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$.

57. A gun is mounted on a gun-carriage, movable on a smooth horizontal plane and the gun is elevated at the angle α to the horizon. A shot is fired and leaves the gun in a direction inclined at an angle θ to the horizon. If the mass of the gun and the carriage be n times that of shot, show that

$$\tan\theta = \left(1 + \frac{1}{n}\right)\tan\alpha$$

58. Two spheres of masses m_1 and m_2 moving with velocities u and v, impinge directly. Show that there is always loss of kinetic energy unless the elasticity is perfect.

59. State and prove the theorem of parallel axes for the moment of inertia of a rigid body.

60. Find the moment of inertia of an elliptic disc

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ (a > b)$$

about its major axis.

(*Turn Over*)

GROUP—C

(For Pass students)

Course No. : MTMDSE–501T (P)

(MATRICES)

SECTION—A

Answer any *twenty* of the following as directed : $1\times20=20$

1. When is a set of vectors said to be linearly independent?

2. What is zero vector?

3. Write down the standard basis of R^3.

4. Are the vectors $\alpha+\beta$, $\alpha-\beta$, $\alpha-2\beta+\gamma$ linearly independent?

5. Is the set of vectors

$$\alpha = \{(a_1, a_2, a_3) \in R^3 / a_1 \geq 0\}$$

a subspace of R^3?

6. What do you mean by linear combination of vectors?

7. If S and T are two subsets of a vector space $V(F)$ and $S \subseteq T$, then what should be the relation between $L(S)$ and $L(T)$?

8. A system consisting of a single non-zero vector is linearly _____.

(Fill in the blank)

9. Define eigenvector of a linear operator.

10. Can a eigenvector of a linear operator $T: V \to V$ has more than one eigenvalue?

11. What do you mean by invariant subspace of a vector space?

12. What is the eigenvalue of an invertible linear operator on a finite dimensional vector space?

13. What should be the standard form of unit matrix when reflection is done through x-axis?

14. If λ is an eigenvalue of a matrix A and matrix B is similar to A, then what is the eigenvalue of B?

15. What is the difference between rotation and translation?

16. What do you mean by eigenspace?

17. Define Hermitian matrix.

18. Give one example of a skew-symmetric matrix.

19. What do you mean by normal form of a matrix?

20. When is an equation said to be homogeneous?

21. Define elementary matrix.

22. When is a matrix said to be upper triangular matrix?

23. Define idempotent matrix.

24. Find the rank of matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

25. Define inverse of a matrix.

26. What is a diagonal matrix?

27. If $|A| = 3$, then what should be value of $|A^{-1}|$?

28. Is the matrix $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ diagonalizable?

29. Is matrix multiplication commutative?

30. What is the inverse of a diagonal matrix?

31. When is a matrix said to be singular?

32. Find the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$.

33. Define rank of a matrix.

34. What is augmented matrix?

35. When is an equation said to be linear?

36. What is the coefficient matrix of the following system of linear equations?

$$3x + 4y + z = 6;\ x + 2y - 3z = 2;\ x + 2y + 5z = 6$$

37. When a system of linear equation has infinite number of solution?

38. When are two matrices said to be similar?

39. Give one example of two non-zero matrices whose product is zero.

40. Can we find inverse of a singular matrix?

SECTION—B

Answer any *five* of the following questions : 2×5=10

41. Express each of the standard basis vectors of R^3 as a linear combination of $\alpha_1 = (1, 0, -1)$; $\alpha_2 = (1, 2, 1)$; $\alpha_3 = (0, -3, 2)$.

42. Show that the vectors (1, 3, 2), (1, −7, −8) and (2, 1, −1) is linearly dependent.

43. If c is the eigenvalue of an invertible transformation T, then show that c^{-1} is the eigenvalue of T^{-1}.

44. If T is any linear operator on a vector space, then show that the null space of T is invariant under T.

45. Show that the rank of the transpose of a matrix is the same as that of original matrix.

46. The point (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear, then show that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \leq 3$$

47. Show that the inverse of a matrix is unique.

48. If $AB = BA$ and $S^2 = B$, then show that $(A^{-1} SA)^2 = B$.

49. When a system of equation $AX = B$ has *(a)* no solution and *(b)* unique solution?

50. Find the determinant of

$$\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$$

where $a^2 + b^2 + c^2 + d^2 = 1$. Also find the cofactor of $a+ib$.

SECTION—C

Answer *any* **five** questions

51. *(a)* Show that the vectors $\alpha_1 = (1, 2, 1)$; $\alpha_2 = (2, 1, 0)$; $\alpha_3 = (1, -1, 2)$ form a basis of R^3. 4

(b) Is the vector (2, –5, 3) in the subspace of R^3 spanned by the vectors (1, –3, 2), (2, –4, –1), (1, –5, 7)? 4

52. *(a)* Select a basis if any three of R^3 (R) form the set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where $\alpha_1 = (1, -3, 2)$; $\alpha_2 = (2, 4, 1)$; $\alpha_3 = (3, 1, 3)$; $\alpha_4 = (1, 1, 1)$. 4

(b) In the vector space R^3, express the vector (1, –2, 5) as a linear combination of vectors (1, 1, 1), (1, 2, 3), (2, –1, 1). 4

53. *(a)* Show that the space generated by (1, 1, 1) and (1, 2, 1) is an invariant subspace of R^3 under T, where

$$T(x, y, z) = (x+y-z, \; x+y, \; x+y-z)$$ 4

(b) Find all eigenvalues and eigenvectors of a matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ 4

54. *(a)* If A and B are n square matrices, then show that AB and BA has the same eigenvalues. 4

(b) If λ is an eigenvalue of A, then show that λ is an eigenvalue of A^T. 4

55. *(a)* Reduce the matrix A to its normal form, where

$$A=\begin{bmatrix}0 & 1 & -3 & -1\\ 1 & 0 & 1 & 1\\ 3 & 1 & 0 & 2\\ 1 & 1 & -2 & 0\end{bmatrix}$$

4

(b) Find the rank of $A+B$, if

$$A=\begin{bmatrix}1 & 5 & 4\\ 0 & 3 & 2\\ 2 & 13 & 10\end{bmatrix},\quad B=\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 3 & 3 & 3\end{bmatrix}.$$

4

56. *(a)* Show that the interchange of two rows (columns) does not alter the rank. 4

(b) Show that every square matrix can be uniquely as a sum of a symmetric and a skew-symmetric matrix. 4

57. *(a)* Let T be a linear operator on R^3 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}$$

Prove that T is diagonalizable. 4

(b) If A, B are non-singular matrices of same order, then show that $(AB)^{-1}=B^{-1}A^{-1}$. 4

58. *(a)* Prove that a necessary and sufficient condition that an $n\times m$ matrix A over a field F be diagonalizable is that A has a linearly independent eigenvector. 4

(b) Find the inverse of the matrix

$$\begin{bmatrix}1 & 2 & 1\\ 3 & 2 & 3\\ 1 & 1 & 2\end{bmatrix}$$

by using elementary row operation. 4

59. *(a)* Is the following system of equations consistent or not?

$$9x+7y+3z=6$$
$$5x-y+4z=1$$
$$3x+5y+z=2$$

If consistent, find its solution. 4

(b) Find the rank of A, where

$$A=\begin{bmatrix}1 & 3 & 4 & 5\\ 1 & 2 & 6 & 7\\ 1 & 5 & 0 & 1\end{bmatrix}$$

4

60. *(a)* Define elementary transformation of a matrix. 4

(b) Deduce the normal form and hence find the rank of the matrix

$$\begin{bmatrix}1 & 1 & 1 & -1\\ 1 & 2 & 3 & 4\\ 3 & 4 & 5 & 2\end{bmatrix}$$

4

GROUP—D

(For Pass students)

Course No. : MTMDSE-501T (P)

(LINEAR ALGEBRA)

SECTION—A

Answer any *twenty* of the following as directed : $1\times20=20$

1. Define vector space.

2. Define subspace of a vector space.

3. Define linear independence.

4. Can two subspaces of a vector space be disjoint? Justify.

5. What is basis?

6. What is the dimension of vector space $\mathbb{R}^4(\mathbb{R})$?

7. If u and v are two LI vectors in $\mathbb{R}^2(\mathbb{R})$, what is the linear span of $\{u, v\}$?

8. Any four vectors in the vector space $\mathbb{R}^3(\mathbb{R})$ are LD.

(State True or False)

9. Define linear transformation.

10. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(x, y) = (x+1, y+1) \forall (x, y) \in R^2$$

Justify, if T is a linear transformation.

11. Let $T: U \to V$ be the linear transformation $T(x) = 0 \forall x \in U$. What is the null space of T?

12. Define nullity of a linear transformation.

13. Define rank of a linear transformation.

14. Let $T: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation such that dim $N(T)$ is 1. What is dim $R(T)$?

15. If $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(0, 1) = (1, 2)$ and $T(1, 0) = (2, 3)$, what is the matrix of T with respect to standard ordered basis of $\mathbb{R}^2$?

16. Define matrix of a linear transformation.

17. If $T: V \to U$ is an isomorphism, then what is the null space of T?

18. Can there be any isomorphism from $\mathbb{R}^4(\mathbb{R})$ to $\mathbb{R}^2(\mathbb{R})$? Justify.

19. If $\ker T = \{0\}$, then T is one-one.

(State True or False)

20. Is the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x, y) = (x + 2y, 2x + 4y) \forall (x, y) \in \mathbb{R}^2$$

an isomorphism?

21. Contract the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ which takes (1, 0) to (1, 3) and (0, 1) to (3, 1).

22. Let V be the vector space of all polynomials $P(x)$ of degree at most 3 with real coefficients. Let $T: V \to V$ be defined by

$$T(P(x)) = \frac{d}{dx} P(x)$$

Find $T(1 + x + x^2)$.

23. Define idempotent operator.

24. Define nilpotent operator.

25. Define eigenvalue.

26. Define eigenvector.

27. Define eigenspace.

28. What are the eigenvalues of a diagonal matrix?

29. How many distinct eigenvalues does a null matrix of order n have?

30. If the characteristic polynomial of a matrix A is $\lambda^2 - 1$, then what is A^2?

31. If 0 is an eigenvalue of A, then the system of linear equations $Ax = 0$ has non-trivial solutions.

(State True or False)

32. Let A be a 3×3 matrix with distinct eigenvalues 1, 2, 3. What is the characteristic equation of A?

33. Define inner product space.

34. Define Euclidean space.

35. Define unitary space.

36. Prove that $< 0, u > = 0 \; \forall u \in V$, where V is a vector space over F.

37. If x and y are orthogonal unit vectors in an inner product space, what is the value of $< x + iy, y >$?

38. If x and y are vectors in a complex inner product space such that $< ix, y > = < x, iy >$, show that x and y are orthogonal.

39. If u and v are vectors in an inner product space such that $||u|| = 3$, $||v|| = 2$, then what is the maximum value of $|< u, v >|$?

40. Let x, y be orthogonal vectors in an inner product space such that $||x|| = 2$, $||y|| = 3$, what is $||x + y||$?

SECTION—B

Answer any *five* of the following questions : 2×5=10

41. Let $V(F)$ be a vector space. Let $x \in V$ and $\alpha \in F$ be arbitrary. Show that

$$\alpha(-x) = (-\alpha)x = -(\alpha x)$$

42. Show that the union of two spaces of a vector space may not be a vector space.

43. Let $T: V \to W$ be a linear transformation, then prove that

$$T(v_1 - v_2) = T(v_1) - T(v_2)$$

44. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear $T(1, 0) = (1, 4)$ and $T(1, 1) = (2, 4)$. What is $T(2, 3)$?

45. Reduce the matrix in echelon form

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \end{bmatrix}$$

46. Investigate for what values of λ and μ the equations

$$x + y + z = 6, \; x + 2y + 3z = 10, \; x + 2y + \lambda z = \mu$$

have no solution.

47. Let T be an invertible linear operator on a finite dimensional vector space V over F. Prove that $\lambda \in F$ is a characteristic root of T iff λ^{-1} is a characteristic root of T^{-1}.

48. Find the characteristic polynomial for the matrix

$$A = \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$$

where a, b, c are real.

49. Let $u = (1, 2)$, $v = (-1, 1)$ in $\mathbb{R}^2(\mathbb{R})$. If w is a vector in $\mathbb{R}^2$ such that $\langle u, w \rangle = -1$ and $\langle v, w \rangle = 3$, find w.

50. Show that for any vector $v \in \mathbb{R}^2(\mathbb{R})$

$$v = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2$$

where $e_1 = (1, 0)$, $e_2 = (0, 1)$.

SECTION—C

Answer *any* **five** questions

51. *(a)* Show that the intersection of an arbitrary collection of subspaces of a vector space V is also a subspace of V. 4

(b) If W_1 and W_2 are subspaces of a finite dimensional vector space $V(F)$, then show that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \quad 4$$

52. *(a)* State and prove extension theorem. 4

(b) Prove that in a vector space a set of vectors $s = \{x_1, x_2, \cdots, x_n\}$ is LD, iff some elements of s is a linear combination of others. 4

53. (a) If V is a vector space and $T: V \to V$ is a linear operator, prove that the following are equivalent : 4

(i) Range $(T) \cap \ker(T) = \{0\}$

(ii) $T(T_x) = 0 \Rightarrow T_x = 0$

(b) Find range, rank, kernal and nullity of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x, y) = (x + y, x)$$ 4

54. (a) Find the matrix of $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$T(x, y, z) = (x - y + z, 2x + 3y - 2z, x + y - 2z)$

relative to the basis

$B_1 = \{(1, 1, 0), (5, -1, 2), (1, 2, 1)\}$

$B_2 = \{(1, 1, 0), (0, 0, 1), (1, 5, 2)\}$ 4

(b) State and prove Sylvester's law of nullity. 4

55. (a) Let T_1 and T_2 be two linear transformations from $V(F)$ to $W(F)$, then show that $T_1 + T_2$ is a linear transformation. 4

(b) Reduce the matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

to echelon form and hence find the rank of A. 4

56. (a) Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. Prove that

rank $(AB) \le \min\{\text{rank } A, \text{rank } B\}$ 4

(b) Determine the conditions for which the system of equation admits of—

(i) no solution;

(ii) many solutions. 4

57. (a) Show that eigenvalues of

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

are ± 1. 3

(b) State and prove Cayley-Hamilton theorem. 5

58. (a) If λ is an eigenvalues of the matrix A, then prove that 0 is an eigenvalues of $A - \lambda I_n$. 3

(b) Given

$$A = \begin{pmatrix} 3 & 1 & 4 \\ 6 & 5 & -1 \\ 3 & 1 & 0 \end{pmatrix}.$$

Verify Cayley-Hamilton theorem and find A^{-1}. 5

59. (a) State and prove Cauchy-Schwarz inequality in an inner product space. 4

(b) Prove that for all $x, y \in V$

$$||x+y||^2 + ||x-y||^2 = 2\{||x||^2 + ||y||^2\}$$ 4

60. (a) State and prove Bessel's inequality in an inner product space. 5

(b) Let V be an inner product space over F. Prove that for all $x, y \in V$

$$< x, y > = \frac{1}{4}||x+y||^2 - \frac{1}{4}||x-y||^2$$ 3

★ ★ ★