TDC (CBCS) Even Semester Exam., 2022

MATHEMATICS

(4th Semester)

Course No. : MTMDSC/GEC–401

(Abstract Algebra)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *twenty* of the following questions : 1×20=20

1. Define group.

2. Give an example of a non-Abelian group.

3. What is the identity element of the group $(\mathbb{Z}, *)$, where $a * b = a + b + 2$?

4. Find the inverse of a^{-1}, where a is an element of a group G.

5. Is the set of irrational numbers a group under addition? Justify.

6. Write the generators of the group $(\mathbb{Z}, +)$.

7. Give an example of a commutative group which is not cyclic.

8. Define subgroup of a group.

9. What is the centre $Z(G)$ of an Abelian group G?

10. Let G be a cyclic group of infinite order. Find the number of elements of finite order in G.

11. Define order of an element of a group.

12. What is the number of elements of order 5 in the cyclic group of order 25?

13. Let H and K be two subgroups of a group G such that H has 7 elements and K has 13 elements. What is the number of elements in HK?

14. Let G be a group of order 25. Is there any subgroup of order 3?

15. Give an example to show that a left coset of a subgroup of a group is not equal to a right coset of that subgroup.

16. Define normal subgroup of a group.

17. What is the order of the quotient group $\mathbb{Z}/10\mathbb{Z}$?

18. Find the number of isomorphisms from the group $(\mathbb{Z}, +)$ onto the group $(\mathbb{Z}, +)$.

19. Is it true that a non-commutative group may be a homomorphic image of a commutative group?

20. Define kernel of a homomorphism.

21. Define skew field.

22. Give an example of an integral domain which is not a field.

23. Define zero divisor.

24. State the condition under which any integral domain will be a field.

25. Give an example of a commutative ring without identity.

SECTION—B

Answer any *five* of the following questions : 2×5=10

26. Prove that in a group G, for all $a, b \in G$, $(ab)^{-1} = b^{-1}a^{-1}$.

27. Write down the Cayley table for the group operation of the group $\mathbb{Z}_5$.

28. Show that union of two subgroups may not be a subgroup.

29. Prove that every cyclic group is Abelian.

30. Find all distinct left cosets of the subgroup $7\mathbb{Z}$ in the group $\mathbb{Z}$.

31. Give an example of a group G and a subgroup H of G, such that $aH = bH$, but $Ha \neq Hb$ for some $a, b \in G$.

32. If $f : G \to G'$ is a homomorphism, then prove that $f(e) = e'$, where e, e' are the identities of G and G' respectively.

33. Let $\mathbb{R}^+$ be the group of positive real numbers under multiplication and $\mathbb{R}$, the group of all real numbers under addition. Prove that the map $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \log x$ is a homomorphism.

34. Let R be a commutative ring. Prove that $(a+b)^2 = a^2 + 2ab + b^2$ for all $a, b \in R$.

35. Prove that a field is an integral domain.

SECTION—C

Answer any *five* of the following questions : 8×5=40

36. *(a)* Prove that a finite semigroup in which cancellation laws hold is a group. 4

(b) Show that the group $GL(2, \mathbb{R})$ is non-Abelian. 4

37. *(a)* Let $G = \{a \in \mathbb{R} : -1 < a < 1\}$. Define a binary operation $*$ on G by $a * b = \dfrac{a+b}{1+ab}$ for all $a, b \in G$. Show that $(G, *)$ is a group. 4

(b) Write all complex roots of $x^6 = 1$. Show that they form a group under complex multiplication. 4

38. *(a)* Define centre of a group. Prove that centre of a group G is a subgroup of G. 1+3=4

(b) Prove that a subgroup of a cyclic group is cyclic. 4

39. *(a)* Show that a group of finite composite order has at least one non-trivial subgroup. 4

(b) Prove that order of a cyclic group is equal to the order of its generator. 4

40. *(a)* State and prove Lagrange's theorem for finite group. 4

(b) Prove that in a finite group, order of each element divides the order of the group. 4

41. *(a)* Let H be a non-empty subset of a group G. Define $H^{-1} = \{h^{-1} \in G : h \in H\}$. Show that—

(i) if H is a subgroup of G, then $H = H^{-1}$;

(ii) if H, K are subgroups of G, then $(HK)^{-1} = K^{-1}H^{-1}$. 2+2=4

(b) If H and K be two subgroups of a group G, prove that HK is a subgroup of G if and only if $HK = KH$. 4

42. *(a)* Show that a subgroup of index 2 in a group G is a normal subgroup of G. 4

(b) If G is a group such that $G/Z(G)$ is cyclic, where $Z(G)$ is centre of G, then show that G is Abelian. 4

43. *(a)* Show that every subgroup of an Abelian group is normal but the converse is not true. Justify. 4

(b) If $f : G \to G'$ be an onto homomorphism with $K = \ker f$, then prove that $G/K \cong G'$. 4

44. *(a)* Prove that a commutative ring R is an integral domain if and only if for all $a, b, c \in R$ $(a \neq 0)$, $ab = ac \Rightarrow b = c$. 4

(b) Let R be a commutative ring with unity. Show that—

(i) a is a unit if and only if a^{-1} is a unit;

(ii) a, b are units if and only if ab is a unit. 4

45. *(a)* Prove that the set $\mathbb{Z}[i] = \{a + ib | a, b \in \mathbb{Z}\}$ is a ring under the usual addition and multiplication. 4

(b) Show that a non-zero finite integral domain is a field. 4

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