2020/TDC(CBCS)/ODD/SEM/
MTMDSC/GE–301T/333

TDC (CBCS) Odd Semester Exam., 2020
held in March, 2021

MATHEMATICS

(3rd Semester)

Course No. : MTMDSC/MTMGE–301T

(Real Analysis)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *twenty* questions : 1×20=20

1. Define upper bound of a set in $\mathbb{R}$.

2. Find a lower bound of a set of positive rational numbers.

3. Define finite set with an example.

4. Justify if the set

$$A = \{x \in \mathbb{R} \mid x^{2021} + x + 2 = 0\}$$

is finite.

5. Justify if the set

$$A = \{x \in \mathbb{Q} \mid x^2 > 2\}$$

is countable.

6. Construct an onto map from the set N of natural numbers to the set {2020, 2021, 2022}.

7. Give one example of an uncountable but bounded set.

8. What is the least upper bound of $\{1 + \frac{1}{n} \mid n \in \mathbb{N}\}$?

9. Define open set in R.

10. Justify if the set

$$A = \{x \in \mathbb{R} \mid x^2 - 7x + 12 < 0\}$$

is open.

11. Justify if the set Q of rational numbers is closed.

12. What are the derived sets of N and Z?

13. Give example of a set that is neither closed nor open in R.

14. Justify if 5 is a limit point of the set {1, 2, 3, 4, 5, 6}.

15. Define closure of a set in R.

16. Justify if the set {199, 499, 999} is closed in R.

17. Give one example of a divergent sequence.

18. What is the limit of the sequence $\langle x_n \rangle$, where

$$x_n = \frac{(n+1)(2n+3)}{n^2+5}?$$

19. Define limit of a convergent sequence.

20. Write True or False :

$\left\langle \frac{1}{n} \right\rangle$ is an increasing sequence.

21. Give example of a sequence that is bounded above but not below.

22. State monotone convergence theorem.

23. Give example of a sequence that is neither increasing nor decreasing.

24. Give example of a bounded sequence that is not convergent.

25. Give example of a divergent series.

26. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{3^n}$$

27. Give example of a convergent series that is not absolutely convergent.

28. Justify whether the series

$$\sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)^2$$

is convergent.

29. If Σx_n is convergent series, then evaluate

$$\underset{n\to\infty}{\text{Lt}} (2x_n - 3)$$

30. If Σx_n and Σy_n are divergent, can we say that $\Sigma(x_n + y_n)$ is divergent? Justify.

31. State D'Alembert's ratio test.

32. Define convergence of an infinite series.

33. Give the sequential definition of continuity of a function at a point.

34. Use sequential criterion to show that

$$\underset{x\to 0}{\text{Lt}} \frac{1}{x}$$

does not exist.

35. Examine the existence of

$$\lim_{x \to 0} \sin \frac{1}{x}$$

36. If f is continuous at $x = a$ but g is discontinuous at $x = a$, then what can you say about the continuity of $f + g$ at a?

37. Give example of a function that is discontinuous at exactly two points.

38. Find the value of k for which the function

$$f(x) = \begin{cases} x^2 - 2, & x \geq 1 \\ kx + 3, & x < 1 \end{cases}$$

is continuous at $x = 1$.

39. Give example of a function that is discontinuous at every point in $\mathbb{R}$.

40. Write True or False :

A function f is continuous at $x = a$, if

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

(*Continued*)

SECTION—B

Answer any *five* questions : 2×5=10

41. Show that $\mathbb{N} \times \mathbb{N}$ is countable.

42. If A and B are bounded above sets with $A \subseteq B$, then show that $\text{lub}\, A \leq \text{lub}\, B$.

43. Show that b is a limit point of (a, b).

44. Show that every open set in $\mathbb{R}$ is a union of open intervals.

45. Show that the sequence $\langle \sin \frac{1}{n} \rangle$ converges to 0.

46. Evaluate :

$$\lim_{n \to \infty} \frac{1 + 2 + 3 + \cdots + n}{n^2}$$

47. Examine the convergence of Σx_n, where

$$x_n = \frac{n + 3}{n^3 + 3n + 8}$$

(*Turn Over*)

48. Test the convergence of

$$\sum \frac{x^n}{\lfloor n}, x > 0$$

49. Use sequential criterion to test the continuity of the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$

at $x = \sqrt{2}$.

50. If f and g are continuous at $x = c$, then show that $f + g$ is continuous at $x = c$.

SECTION—C

Answer *any* **five** questions

51. *(a)* Show that the union of two countable sets is countable. 4

(b) Use the LUB axiom to prove the Archimedean property of $\mathbb{R}$. 4

52. *(a)* Show that the set of rational numbers is countable. 5

(b) Find the least upper bound and the greatest lower bound of the set

$$\left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$$

Justify your answer. 3

53. *(a)* Show that the arbitrary union of open sets is open. Is the same true for intersection? Justify. 4+1=5

(b) Show that the finite union of closed sets is closed. 3

54. *(a)* Show that a set is closed if and only if it contains all its limit points. 5

(b) Show that the intersection of two open sets is open. 3

55. *(a)* Show that the limit of a sequence, if it exists, is unique. 3

(b) Show that a monotonic increasing sequence bounded above is convergent. 5

56. *(a)* If $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences converging to x and y respectively, then show that $\langle x_n + y_n \rangle$ converges to $x+y$. 4

(b) Show that the sequence $\langle S_n \rangle$, where

$$S_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

is convergent. 4

57. Test the convergence of the series : 2+2+4=8

(i) $\sum \frac{n+1}{n+3}$

(ii) $\frac{1}{1\times2\times3} + \frac{3}{2\times3\times4} + \frac{5}{3\times4\times5} + \cdots$

(iii) $\frac{x}{\sqrt{1\times2}} + \frac{x^2}{\sqrt{2\times3}} + \frac{x^3}{\sqrt{3\times4}} + \cdots \; (x>0)$

58. *(a)* State Cauchy's nth root test and use it to test the convergence of Σx_n, where

$$x_n = \left(1+\frac{1}{n}\right)^{-n^2}$$

1+3=4

(b) Test the convergence of

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

4

59. *(a)* Prove the equivalence of ε-δ definition and the sequential definition of continuity of a function at a point. 5

(b) If f and g are continuous at $x=c$, then show that the product fg is also continuous at $x=c$. 3

60. *(a)* Show that the function

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous at $x=0$. 4

(b) Show that the function

$$f(x) = \frac{1}{1+|x|}, \; x \in \mathbb{R}$$

is bounded. Find its supremum and infimum. 4

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