TDC (CBCS) Odd Semester Exam., 2022

MATHEMATICS

(1st Semester)

Course No. : MTMDSC/MTMGE–101T

(Differential Calculus)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *four* of the following questions : 1×4=4

(a) What is the value of $\lim\limits_{x \to 0} (1+x)^{\frac{1}{x}}$?

(b) Does

$$\lim_{x \to 0} \left\{ \sin\frac{1}{x} + x\sin\frac{1}{x} + x^2 \sin\frac{1}{x} \right\}$$

exist?

(c) Give an example of a function $f(x)$ such that $\lim_{x\to 0} f(x)$ and $f(0)$ exist and are equal.

(d) If $\lim_{x\to 0} \frac{e^x - 1}{x} = k - 1$, then find k.

(e) Does $\lim_{x\to\infty} \frac{1}{x^2}$ exist?

2. Answer any *one* of the following questions : 2

(a) State Cauchy's criterion for the existence of limit of a function.

(b) By using Cauchy's criterion, show that

$$\lim_{x\to 0} \sin\frac{1}{x}$$

does not exist.

Answer *either* Q.No. **3.** *or* **4.** :

3. (a) Using ε-δ definition of limit, evaluate

$$\lim_{x\to 0} x^2 \cos\frac{1}{x}.$$ 4

(b) Find $\lim_{n\to\infty} \frac{x^n - 1}{x^n + 1}$. 4

4. (a) Evaluate

$$\lim_{x\to 0} \frac{\operatorname{cosec} x - \cot x}{x}$$ 2

(b) Examine, if

$$\lim_{x\to 0} \frac{e^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}$$

exists or not. 3

(c) Show that $\lim_{x\to 0} [x]$ does not exist. 3

UNIT—II

5. Answer any *four* of the following questions : 1×4=4

(a) Is the function $f(x) = x^3 + 2x^2 + 1$ continuous at $x = 0$?

(b) For what values of x, $f(x) = \frac{1}{x}$ is continuous?

(c) Show that the derivative of an even function is odd.

(d) Find $\frac{d}{dx}\{\log(\operatorname{cosec} x - \cot x)\}$.

(e) Is the function $f(x) = \cos\frac{1}{x}$ continuous at $x = 0$?

6. Answer any *one* of the following questions : 2

(a) If a function f is continuous at a point $x = a$, then prove that $|f|$ is also continuous at $x = a$.

(b) If a function f is differentiable at a point $x = a$, prove that f is continuous at $x = a$.

Answer *either* Q.No. **7.** *or* **8.** :

7. (a) Show that the function $f(x) = |x-2|$ is not differentiable at $x = 1$, though it is continuous there. 4

(b) Prove that if $f(x)$ is continuous at $x = a$ and $f(a) \neq 0$, then $f(x)$ has the same sign at that of $f(a)$ in a neighbourhood of $x = a$. 4

8. (a) Let

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that $f(x)$ is continuous at $x = 0$. Also examine the differentiability of $f(x)$ at $x = 0$. 4

(b) Let f be a function such that for all real values of x and y, $f(x+y) = f(x) + f(y)$. If f is continuous at $x = a$, then prove that f is continuous for all real values of x. 4

UNIT—III

9. Answer any *four* of the following questions : 1×4=4

(a) Find y_n, if $y = x^n$.

(b) If $y = \sin^{-1} x$, then show that $(1-x^2)y_2 - xy_1 = 0$.

(c) If $u = x\sin y + y\sin x$, find $\frac{\partial^2 u}{\partial x \partial y}$.

(d) Define a homogeneous function of degree n in two variables.

(e) Is the function $y^2 \log\left(\frac{x}{y}\right)$ a homogeneous function? If so, find its degree.

10. Answer any *one* of the following questions : 2

(a) Find the nth derivative of $x^{n-1} \log x$.

(b) State Euler's theorem on homogeneous function of degree n in two variables.

Answer *either* Q.No. **11.** *or* **12.** :

11. (a) State and prove Leibnitz's theorem on successive differentiation. 4

(b) If

$$u = \tan^{-1}\left(\frac{x^3 - y^3}{x - y}\right)$$

prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$ 4

12. (a) If $y = x^{2n}$, $n \in \mathbb{N}$, show that

$$y_n = 2^n \{1 \cdot 3 \cdot 5 \cdots (2n-1)\} x^n$$ 3

(b) If $y = \sin^{-1} x$, show that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$$ 5

UNIT—IV

13. Answer any *four* of the following questions : 1×4=4

(a) What is the geometrical meaning of $\frac{dy}{dx}$ at a point?

(b) Find the slope of the curve $y = x^2$ at the point (1, 1).

(c) What is the condition that the two curves $\phi(x, y) = 0$ and $\psi(x, y) = 0$ cut orthogonally?

(d) Define radius of curvature at any point on a curve.

(e) About which axis the curve $x^2 = 2y$ is symmetrical?

14. Answer any *one* of the following questions : 2

(a) Find the point where two tangents to the curve $y = x^3 - 3x^2 - 9x + 15$ is parallel to x-axis.

(b) Find the polar subtangent for the curve $\frac{l}{r} = 1 + e\cos\theta$.

Answer *either* Q.No. **15.** *or* **16.** :

15. (a) What do you mean by angle between two curves? Prove that the curves

$$\frac{x^2}{a} + \frac{y^2}{b} = 1 \text{ and } \frac{x^2}{a'} + \frac{y^2}{b'} = 1$$

will cut orthogonally if $a - b = a' - b'$. 1+3=4

(b) Find the radius of curvature at any point (s, ψ) on the curve $s = 4a\sin\psi$. Also, show that for any curve

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2$$ 1+3=4

16. (a) Show that at any point on the curve $x^{m+n} = k^{m-n} y^{2n}$, the mth power of the subtangent varies as the nth power of the subnormal. 4

(b) Find the radius of curvature for the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = 0$. 4

UNIT—V

17. Answer any *four* of the following questions : 1×4=4

(a) Give an example of a function whose maximum value is less than its minimum value.

(b) Write down Cauchy's form of remainder in Taylor's theorem.

(c) Evaluate

$$\lim_{x \to 0} \frac{x - \sin x}{x^2}$$

(d) State Lagrange's mean value theorem.

(e) Write $\cos x$ in ascending powers of x.

18. Answer any *one* of the following questions : 2

(a) Give the geometrical meaning of Rolle's theorem.

(b) In the mean value theorem

$$f(a+h) = f(a) + hf'(a + \theta h)$$

if $a = 1$, $h = 3$, $f(x) = \sqrt{x}$, find the value of θ.

Answer *either* Q.No. **19.** *or* **20.** :

19. (a) Evaluate

$$\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{\frac{1}{x^2}}$$

4

(b) Show that the largest rectangle (in respect of area) inscribed in a circle is a square. 4

20. (a) Evaluate : 2+3=5

(i) $\lim_{x \to 1} \left(\frac{1}{x^2 - 1} - \frac{2}{x^4 - 1} \right)$

(ii) $\lim_{x \to 0} (\cos x)^{\cot^2 x}$

(b) Show that the maximum value of $x^{\frac{1}{x}}$ is $e^{\frac{1}{e}}$. 3

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