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TDC (CBCS) Odd Semester Exam., 2021 held in March, 2022

MATHEMATICS

(5th Semester)

Course No. : MATHCC–501T

(Topology)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following questions : 2×10=20

1. Define bounded metric space and give an example.

2. Show that in a discrete metric space every set is open.

3. Let X be a metric space and let G be an open set in X. Prove that $G \cap A = \phi$ if and only if $G \cap \overline{A} = \phi$.

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4. Show that every convergent sequence in a metric space is a Cauchy sequence.

5. Give an example of a metric space which is not complete.

6. Define continuity of a function in metric space.

7. Define a topological space and give an example.

8. Define discrete and indiscrete topologies.

9. Write two distinct topologies on $X = \{a, b, c\}$.

10. Define metrizable space.

11. Show with an example that the union of two topologies on a set may not be a topology.

12. Define interior and exterior points of a set in a topological space.

13. Find the condition that a function f from topological spaces X to a topological space Y is not continuous at a point $x \in X$.

14. Define homeomorphism of a function in topological space.

15. Show that identity function is continuous.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. *(a)* Let X be a non-empty set. Show that the function $d: X \times X \to \mathbb{R}$ is a metric on X if and only if d satisfies the conditions—

(i) $d(x, y) = 0 \Leftrightarrow x = y \; \forall \; x, y \in X$;

(ii) $d(x, y) \le d(x, z) + d(y, z) \; \forall \; x, y, z \in X$. 5

(b) If (X, d) be a metric space and A is a subset of X, then show that $\overline{A}$ is the smallest closed set containing A. 5

17. *(a)* Let $S(x_0, r)$ be an open sphere in a metric space (X, d). Prove that to each $p \in S(x_0, r)$ there exists $r' > 0$ such that $S(p, r') \subseteq S(x_0, r)$. 5

(b) Let A be a subset of a metric space (X, d). Prove that $\overline{A} = A \cup D(A)$. 5

18. *(a)* If x and y are two points of a metric space (X, d) such that the sequences $< x_n >$ and $< y_n >$ in (X, d) converges to x and y respectively, then prove that the sequence $< d(x_n, y_n) >$ converges to $d(x, y)$. 5

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(b) Let (X, d) and (Y, ρ) be metric spaces. Show that a function $f: X \to Y$ is continuous if and only if for every subset $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$. 5

19. (a) Show that $\mathbb{R}$ (with the usual metric) is a complete metric space. 5

(b) Let (X, d) and (Y, ρ) be metric spaces and $f: X \to Y$ a mapping. If f is continuous at $x \in X$, then show that for every open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing x such that $f(U) \subseteq V$. 5

20. (a) Let $\mathbb{N}$ be the set of all natural numbers and T the family of subsets of $\mathbb{N}$ consisting ϕ and the sets of the form

$$T_n = \{n, n+1, n+2, \cdots\}, n \in \mathbb{N}$$

Prove that T is a topology for $\mathbb{N}$. 5

(b) Let $\mathbb{R}$ be the set of all real numbers and T the collection of all those subsets S of $\mathbb{R}$ such that either $S = \phi$ or $S \neq \phi$, then for each $x \in S$ there exists a right half open interval H such that $x \in H \subseteq S$. Prove that T is a topology for $\mathbb{R}$. 5

21. (a) Show that an arbitrary intersection of closed subsets of a topological space is a closed set. 5

(b) Define upper limit topology on $\mathbb{R}$. Establish that it is a topology. 5

22. (a) Show that the intersection of arbitrary collection of topologies on a set is also a topology. 5

(b) Show that every metric space is a topological space. 5

23. (a) Let (X, T) be a topological space and $A \subseteq X$. Show that $\text{int}(A)$ is the largest open subset of X containing A. 5

(b) Let A and B be any two subsets of a topological space. Prove that—

(i) $A \subseteq B \Rightarrow D(A) \subseteq D(B)$;

(ii) $D(A \cup B) = D(A) \cup D(B)$. 2+3=5

24. (a) Let X and Y be topological spaces. Show that a function $f: X \to Y$ is continuous iff the inverse image under f of every open set in Y is open in X. 5

$[a_1, b_1) \cap [a_2, b_2)$.

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(b) Let T_1 and T_2 denote the discrete and usual topology respectively on $\mathbb{R}$. Show that the function $f:(\mathbb{R}, T_1) \to (\mathbb{R}, T_2)$ defined by

$$f(x) = \begin{cases} 1 & \text{if } \quad x \in Q \\ 0 & \text{if } \quad x \in \mathbb{R} - Q \end{cases}$$

is continuous. 5

25. (a) Prove that $f: X \to Y$ is a homomorphism if and only if f is both continuous and open. 5

(b) Prove that a constant function from one topological space to another is continuous. 5

$f(x) \in G$

~~$S_\delta(f(x)) \subseteq G$~~

$S(f(x), \varepsilon) \subseteq G$

Again $f(S_\delta(x)) \subseteq S_\varepsilon(f(x))$

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