



**2021/TDC/CBCS/ODD/  
MATDSE-501T (A/B/C/D)/331**

**TDC (CBCS) Odd Semester Exam., 2021  
held in March, 2022**

**MATHEMATICS**

**( 5th Semester )**

**Course No. : MATDSE-501T**

Full Marks : 70

Pass Marks : 28

**Time : 3 hours**

*The figures in the margin indicate full marks  
for the questions*

Honours students should answer *either* from  
Option—A or Option—B

Pass students should answer *either* from  
Option—C or Option—D

**OPTION—A**

**( For Honours students only )**

**Course No. : MATDSE-501T (A)**

**( Mechanics )**

**SECTION—A**

Answer any *twenty* of the following as directed :

1×20=20

1. Any system of coplanar forces acting on a rigid body can be reduced ultimately to a



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single force or a single couple, unless it is in \_\_\_\_.

( Fill in the blank )

2. If three coplanar forces acting on a rigid body be in equilibrium, they must either all three \_\_\_\_ at a point or else all must be \_\_\_\_ to one another.

( Fill in the blanks )

3. Define coefficient of friction.

4. What do you mean by angle of friction?

5. Define cone of friction.

6. Define angular velocity.

7. Write the expressions for radial and cross-radial (transverse) components of acceleration.

8. Define SHM.

9. Where do you find minimum and maximum velocities of a particle executing SHM?

10. Time period of SHM is independent of the amplitude.

( Write True or False )

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( Continued )

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11. Write the equation of motion when the law of attraction is  $\frac{\mu}{x^{5/3}}$ .

12. Write the expression of velocity for a particle moving in an elliptical orbit.

13. What do you mean by areal velocity?

14. If velocity of a particle is given by

$$v^2 = \frac{2\mu}{r}$$

is the orbit hyperbolic? If not, what can you say about the orbit?

15. If a particle of mass  $m$  falls from rest under gravity (supposed constant) in a medium, whose resistance varies as the square of the velocity, write the equation of motion.

16. Define kinetic energy.

17. Define potential energy.

18. State the principle of work.

19. What is Newton's experimental law of direct impact?

20. If the coefficient of elasticity  $e$  is zero, the body is said to be \_\_\_\_ and if  $e = 1$ , the body is said to be \_\_\_\_.

( Fill in the blanks )

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( Turn Over )



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- ✓ 21. Define moment of inertia of a particle about a line.
- ✓ 22. State perpendicular axes theorem.
23. State parallel axes theorem.
24. If the moments and product of inertia about three perpendicular axes are  $A, B, C$  and  $D, E, F$ , what is the moment of inertia about a line through their meeting point?
- ✓ 25. State D'Alembert's principle.

SECTION—B

Answer any five of the following questions :  $2 \times 5 = 10$

26. Write any two laws of limiting friction.
27. State a set of necessary and sufficient conditions for the equilibrium of a system of coplanar forces acting on a rigid body.
28. The velocities of a particle along and perpendicular to the radius vector are respectively  $\lambda_r$  and  $\mu_\theta$  from a fixed origin. Find the path.

- ✓ 29. If the tangential and normal accelerations of a particle describing a plane curve be constant throughout, prove that the radius of curvature at any point is given by  $\rho = (at + b)^2$ .
- ✓ 30. State Kepler's laws of planetary motion.
31. If a particle is projected upwards against gravity (supposed constant) in a resisting medium whose resistance varies as the velocity, obtain the expression for terminal velocity in the medium.
- ✓ 32. State the principle of conservation of momentum.
- ✓ 33. Prove that, impulse = change in momentum.
34. Show that the moment of inertia of a body about a line can be expressed in the form
$$I = \sum m(\vec{r} \times \hat{a})^2$$
where  $\hat{a}$  is the unit vector in the direction of the line, the origin being taken on the line.
- ✓ 35. What is the MI of a circular ring of radius  $a$  and mass  $M$  (a) about a diameter and (b) about an axis through its centre and perpendicular to its plane?



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SECTION—C

Answer any five of the following questions : 8×5=40

36. (a) A uniform beam of length  $2a$  rests in equilibrium against a smooth vertical wall and upon a peg at a distance  $b$  from the wall. Show that the inclination of the beam to the vertical is

$$\sin^{-1}\left(\frac{b}{a}\right)^{1/3}$$

4

- (b) Forces  $P, Q, R$  and  $S$  act along the sides  $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$  and  $\overrightarrow{DA}$  of a cyclic quadrilateral  $ABCD$ , taken in order, where  $A$  and  $B$  are the extremities of a diameter. If they are in equilibrium, then prove that

$$R^2 = P^2 + Q^2 + S^2 + \frac{2PQS}{R}$$

4

37. (a) A uniform ladder rests in limiting equilibrium with its lower end on a rough horizontal plane and its upper end against a smooth vertical wall. If  $\theta$  be the inclination of the ladder to the vertical, then prove that  $\tan \theta = 2\mu$ , where  $\mu$  is the coefficient of friction.

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- (b) A heavy uniform rod rests in equilibrium (limiting) within a fixed rough hollow sphere. If  $\lambda$  be the angle of friction and  $2\alpha$  the angle subtended by the rod at the centre of sphere, show that the inclination  $\theta$  of the rod to the horizon is given by

$$2 \tan \theta = \tan(\alpha + \lambda) - \tan(\alpha - \lambda)$$

4

38. (a) The acceleration of a point moving in a plane curve is resolved into two components, one parallel to the initial line and the other along the radius vector; prove that these components are

$$-\frac{1}{r \sin \theta} \frac{d}{dt} (r^2 \dot{\theta})$$

$$\text{and } \frac{\cot \theta}{r} \frac{d}{dt} (r^2 \dot{\theta}) + \ddot{r} - r \dot{\theta}^2$$

4

- (b) A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their intensities being  $\mu$  and  $\mu'$ ; the particle is slightly displaced towards one of them, show that the time of a small oscillation is

$$\frac{2\pi}{\sqrt{\mu + \mu'}}$$

4



39. (a) Obtain the expressions for tangential and normal components of acceleration of a particle moving in a plane curve.

5

(b) A body performing SHM in a straight line OAB has velocity zero when at points A and B whose distances from O are  $a$  and  $b$  respectively and has velocity  $v$  when half-way between them. Show that the complete period is

$$\frac{\pi(b-a)}{v}$$

3

40. (a) A particle moves in a straight line from a distance  $a$  towards the centre of force, the force being  $\frac{\mu}{x^3}$  at a distance  $x$  from the centre. Show that it reaches the centre in time  $\frac{a^2}{\sqrt{\mu}}$ .

5

(b) If  $v_1$  and  $v_2$  are the linear velocities of a planet when it is respectively nearest and farthest from the sun, prove that  $(1-e)v_1 = (1+e)v_2$

3

41. (a) A planet is describing an ellipse about the sun as focus, show that its velocity away from the sun is greatest when the

radius vector to the planet is at right angles to the major axis of the path and that it then is

$$\frac{2\pi a e T}{\sqrt{1-e^2}}$$

where  $2a$  is the major axis,  $e$  the eccentricity and  $T$  the periodic time.

4

(b) A particle of mass  $m$  is projected vertically under gravity, the resistance of the air being  $mk$  times the velocity. Show that the greatest height attained by the particle is

$$\frac{V^2}{g} [\lambda - \log(1+\lambda)]$$

where  $V$  is the terminal velocity of the particle and  $\lambda V$  is the initial vertical velocity.

4

42. (a) A shell lying in a straight smooth horizontal tube suddenly explodes and breaks into portions of masses  $m$  and  $m'$ . If  $d$  is the distance apart of the masses after a time  $t$ , show that the work done by the explosion is

$$\frac{1}{2} \cdot \frac{mm'}{m+m'} \cdot \frac{d^2}{t^2}$$

5



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43. (b) If two equal and perfectly elastic spheres moving with velocities  $u$  and  $u'$  impinge directly, show that they interchange their velocities after impact.

(a) A sphere impinges directly on an equal sphere at rest. Show that a portion

$$\frac{1}{2}(1-e^2)$$

of the original KE is lost in the impact.

(b) A particle falls from a height  $h$  in time  $t$  upon a fixed horizontal plane. Show that the whole distance (up and down) described by the particle before it has finished rebound is

$$\frac{(1+e^2)}{(1-e^2)} h$$

44. Show that the MI of an elliptic area of mass  $M$  and semi-axes  $a$  and  $b$  about a tangent is  $\frac{5}{4} Mp^2$ , where  $p$  is the perpendicular from the centre on the tangent.

45. A uniform rod  $OA$ , of length  $2a$ , free to turn about its end  $O$ , revolves with uniform angular velocity  $\omega$  about the vertical  $OZ$  through  $O$ , and is inclined at a constant  $\alpha$  to  $OZ$ , show that the value of  $\alpha$  is either zero or

$$\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$$

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OPTION—B

( For Honours students only )

Course No. : MATDSE-501T (B)

( Number Theory )

SECTION—A

Answer any *twenty* of the following questions :

1×20=20

- Write the general solution of the linear diophantine equation  $ax+by=c$ , when  $\text{gcd}(a, b)|c$ .
- How many positive solutions does the equation  $2x+10y=17$  have?
- State Goldbach conjecture.
- If  $ca \equiv cb \pmod{n}$ , is it always true that  $a \equiv b \pmod{n}$ ?
- Find the unit's digit in  $3^{100}$ .
- Calculate  $\tau(180)$ , where  $\tau(n)$  denotes the number of positive divisors of  $n$ .
- State Möbius inversion formula.



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8. Define  $\tau(n)$  and  $\sigma(n)$  in terms of its divisors.
9. Write the value of  $\tau(n)$  in terms of its prime factors.
10. Find  $\sigma(360)$ .
11. Define Euler's phi-function  $\phi(n)$ .
12. Compute  $\phi(16)$ .
13. Can  $3^{40} \equiv 2 \pmod{100}$ ?
14. What is the value of  $\sum_{d|n} \phi(d)$ ?
15. Write the formulae of  $\phi(n)$  in terms of the prime factors of  $n$ .
16. Find the order of 2 modulo 7.
17. Find the number of primitive roots of 9.
18. Does 8 have any primitive root?
19. Does 15 have any primitive root?
20. Define primitive root of an integer.

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21. Define perfect number.
22. Give example of an amicable pair of numbers.
23. Is every Fermat number of the form  $2^{2^n} + 1$  prime for  $n \geq 1$ ?
24. State Pepin's test for primality of Fermat numbers.
25. Write the general solution of  $x^2 + y^2 = z^2$ .

SECTION—B

Answer any five of the following questions :  $2 \times 5 = 10$

26. State Fermat's theorem and Wilson's theorem.
27. Find the remainder when  $41^{65}$  is divided by 7.
28. Show that  $\tau$  is a multiplicative function.
29. Show that  $\tau(n) \leq 2\sqrt{n}$  for any  $n \geq 1$ .
30. For  $n \geq 2$ , show that  $\phi(n)$  is even integer.

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- 31. Show that  $\phi(p^k) = p^k - p^{k-1}$  for any prime  $p$ .
- 32. List all numbers that have primitive root.
- 33. Show that 2 is not a primitive root of  $2^{2^n} + 1$  for any  $n \geq 2$ .
- 34. Show that if  $p$  and  $2p+1$  are primes, then  $2p+1|M_p$  or  $2p+1|M_p+2$  but not both, where  $M_p = 2^p - 1$  is the  $p$ th Mersenne number.
- 35. If  $a^k - 1$  is prime for  $a > 0$  and  $k \geq 2$ , then show that  $a=2$  and  $k$  is a prime.

SECTION—C

Answer any five of the following questions : 8×5=40

- 36. (a) Show that if  $\gcd(a, b) = 1$ , the linear diophantine equation has infinitely many positive solutions. 4
- (b) Prove that  $53^{103} + 103^{53}$  is divisible by 39. 4
- 37. (a) Let  $p$  be a prime and  $a$  be any integer. Show that  $a^p \equiv a \pmod{p}$ . 4
- (b) Solve the congruence  $17x \equiv 9 \pmod{276}$  by Chinese remainder theorem. 4

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- 38. (a) If  $f$  is a multiplicative function, then show that

$$F(n) = \sum_{d|n} f(d)$$

is also multiplicative. 4

- (b) Show that

$$n^{\frac{1}{2}\tau(n)} = \prod_{d|n} d$$

- 39. (a) Derive a formula for  $\sigma(n)$  in terms of the prime factors of  $n$ . 4

- (b) Show that

$$\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n}$$

- 40. (a) If  $\gcd(a, n) = 1$ , then show that

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

- (b) Show that for  $n > 1$ , the sum of positive integers less than  $n$  and relatively prime to  $n$  is  $\frac{1}{2}n\phi(n)$ . 4

- 41. (a) Show that

$$\phi(n) = \left( \sum_{d|n} \frac{\mu(d)}{d} \right) \times n$$

- (b) Show that  $\phi$  is a multiplicative function. 4

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42. (a) If  $p$  is a prime number and  $d|p-1$ , then show that the congruence  $x^d - 1 \equiv 0 \pmod{p}$  has exactly  $d$  solutions.
- (b) If  $m > 2$  and  $n > 2$ , then show that  $mn$  has no primitive roots, whenever  $\gcd(m, n) = 1$ .
43. (a) If  $d|p-1$  and  $p$  is prime, show that there are exactly  $\phi(d)$  incongruent integers having order  $d$  modulo  $p$ .
- (b) For  $k \geq 3$ , show that  $2^k$  has no primitive roots.
44. (a) Show that every even perfect number has the form  $2^{k-1}(2^k - 1)$  where  $k > 1$  and  $2^k - 1$  is prime.
- (b) Show that every even perfect number ends with 6 or 8.
45. (a) If  $p$  is an odd prime, then show that any prime divisor of  $M_p$  is of the form  $2kp + 1$  for  $k \geq 1$ .
- (b) Show that an odd prime  $p$  is a sum of two squares iff  $p \equiv 1 \pmod{4}$ .

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OPTION—C

( For Pass students only )

Course No. : MATDSE-501T (C)

( Matrices )

SECTION—A

Answer any *twenty* of the following questions :

1×20=20

1. Is the set  $\{(1, 2), (3, 6)\}$  a basis of  $\mathbb{R}^2$ ? Justify.
2. Express  $(5, 6)$  as a linear combination of the vectors  $(1, 1)$  and  $(1, 0)$ .
3. Is the set  $V = \{(x, y) | x + y = 2\}$  a subspace of  $\mathbb{R}^2$ ? Justify.
4. What is the dimension of the subspace  $W = \{(x, y, z) | x + y = 0\}$  of  $\mathbb{R}^3$ ?
5. When is a set of vectors in  $\mathbb{R}^3$  said to be linearly independent?
6. What is the reflection of the point  $(3, 2)$  about X-axis?

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7. If the vector  $(1, 2)$  is rotated anticlockwise by  $30^\circ$ , what is the transformed vector?
8. What is the linear transformation that represents reflection about Y-axis?
9. Give one eigenvalue of a singular matrix.
10. What is the linear transformation that represents reflection about the line  $y = x$ ?
11. Define skew-Hermitian matrix.
12. Give example of an orthogonal matrix.
13. Define echelon form of a matrix.
14. What are the three elementary transformations of a matrix?
15. Define idempotent matrix.
16. Define inverse of a matrix.
17. When is a matrix said to be diagonalizable?
18. Show that every diagonal matrix is diagonalizable.

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19. Is the matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 7 & 8 \\ 0 & 0 & 5 \end{pmatrix}$$

diagonalizable? Justify.

20. Let  $A$  be a matrix and  $P$  is a non-singular matrix such that  $PAP^{-1} = \text{diag}(3, 4, -1)$ . What is the determinant of  $A$ ?
21. What is the rank of an identity matrix of order 10?
22.  $A$  is an  $n \times n$  singular matrix. Can it have rank  $n$ ? Justify your answer.
23.  $A$  is a  $5 \times 7$  matrix. Can it have 6 linearly independent columns? Justify your answer.
24. Write the conditions of consistency of a non-homogeneous system of linear equations.
25. Can a system of homogeneous linear equations be inconsistent? Justify your answer.

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SECTION—B

Answer any five of the following questions :  $2 \times 5 = 10$

26. Show that the vectors  $(1, 2, 3)$ ,  $(0, 1, 4)$ ,  $(1, 2, 4)$  are linearly independent.

27. Show that  $\{(1, 1), (0, 1)\}$  is a basis of  $\mathbb{R}^2$ .

28. Find the matrix of transformation representing reflection about X-axis.

29. Define eigenvalue and eigenvector of a matrix.

30. Reduce the matrix A to echelon form :

$$A = \begin{pmatrix} 5 & 7 & 8 & 9 \\ 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

31. If  $x$  and  $y$  are solutions of a linear homogeneous system of equations  $AX=0$ , then show that any linear combination of  $x$  and  $y$  is also a solution of the same system.

32. How can diagonalizability help in computing higher powers of a matrix?

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33. Explain the process of obtaining matrix inverse using elementary row operations.

34. Check the consistency of the system

$$2x + y + 3z = 1$$

$$3x - 2y + z = 7$$

$$5x - y + 4z = 9$$

35. For what values of  $k$  does the following system of equations have non-trivial solution?

$$x + y + z = 0$$

$$2x + 2y + kz = 0$$

$$x + 3y + 3z = 0$$

SECTION—C

Answer any five of the following questions :  $8 \times 5 = 40$

36. (a) If  $V$  and  $W$  are subspaces of  $\mathbb{R}^3$ , then show that  $V \cap W$  is also a subspace of  $\mathbb{R}^3$ . Is the same true for union? Justify.

3+1=4

(b) Show that  $V = \{(x, y) \mid 2x + 3y = 0\}$  is a subspace of  $\mathbb{R}^2$ .

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37. (a) Express (2, 3, 4) as a linear combination of (1, 3, 2), (1, 1, 0) and (1, 2, -2). 5

(b) Show that a set containing a single non-zero vector is linearly independent, whereas {0} is linearly dependent. 3

38. (a) Show that 0 is an eigenvalue of a matrix A if and only if A is singular. 3

(b) Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad 5$$

39. (a) Show that eigenvectors corresponding to distinct eigenvalues of a matrix are linearly independent. 5

(b) Find the characteristic polynomial of

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

Express it as a polynomial in decreasing powers of the indeterminate (i.e., variable say  $\lambda$ ). 3

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40. (a) Define nilpotent matrix. Provide an example. 2

(b) Define idempotent matrix. Provide an example. 2

(c) Solve the homogeneous system of equations : 4

$$\begin{aligned} 2x + 3y - z &= 0 \\ x - y - z + w &= 0 \\ 3x + y - 2z + 3w &= 0 \\ 3x + 2y - 3z + 3w &= 0 \end{aligned}$$

41. (a) Reduce the matrix to normal form : 4

$$\begin{pmatrix} 1 & 3 & 4 & 7 \\ 2 & 1 & 3 & 0 \\ 4 & 7 & 11 & 14 \end{pmatrix}$$

(b) Solve the system of equations using Gaussian elimination : 4

$$\begin{aligned} 2x + y + z &= 7 \\ 3x - y + 3z &= 10 \\ -x + 3y - 2z &= -1 \end{aligned}$$

42. (a) Find the inverse of

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

using elementary row operations. 4



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(b) Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$$

given that two eigenvectors of  $A$  are

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

4

43. (a) Find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

using elementary column operations.

3

(b) Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

5

44. A manufacturer produces three types of products  $A$ ,  $B$  and  $C$ . Each product of type  $A$  requires 12 hours of cutting and 5 hours of finishing. Each product of type  $B$  requires 10 hours of cutting and 3 hours of finishing. Each product of type  $C$  requires 6 hours of cutting and 1 hour of finishing. On a daily basis, the manufacturer has 440 working hours available for cutting and

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120 working hours for finishing. Formulate the problem mathematically using linear equations to determine how many products of each type can be produced so that all the work power is used. Also, check if the resulting system is consistent. Check if it has a unique solution. Write the general solution in case the system is consistent. 3+5=8

45. The measure of the largest angle of a triangle is  $30^\circ$  less than the sum of the other two angles twice the smallest angle is  $9^\circ$  more than the largest angle. Can we determine the measures of all the angles from this information? Formulate the problem as an equivalent system of linear equations. Check if the resulting system is consistent. Check if it has a unique solution. Write the general solution in case the system is consistent. 3+5=8



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OPTION—D

( For Pass students only )

Course No. : MATDSE-501T (D)

( Linear Algebra )

SECTION—A

Answer any *twenty* of the following as directed :  
1×20=20

1. Define linear dependence of vectors.
2. Let  $V(F)$  be a vector space and  $0$  be the zero vector of  $V$ . Then show that  $\alpha 0 = 0, \forall \alpha \in F$ .
3. When is the union of two subspaces of a vector space also a subspace?
4. Prove that every singleton set containing non-zero vector is LI.
5. Define basis of a vector space.
6. Define linear operator.
7. Define kernel of a linear transformation.

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8. Define range space of a linear transformation.
9. If  $V$  is a vector space and  $T : V \rightarrow V$  is a linear operator, then what is the intersection of range of  $T$  and the null space of  $T$ ?
10. State Sylvester's law of nullity.
11. Let  $T_1$  and  $T_2$  be two linear transformations from  $V(F)$  to  $U(F)$ . Then  $T_1 + T_2$  is also a linear transformation.  
( Write True or False )
12. Let  $V(R)$  be the vector space of all complex numbers  $a + ib$  over the field of reals  $R$  and let  $T$  be a mapping from  $V(R)$  to  $V_2(R)$  defined as  $T(a + ib) = (a, b)$ . Then show that  $T$  is onto.
13. Give an example of homomorphism on vector space.
14. Define homomorphism on vector space.
15. Write down the necessary and sufficient condition for a linear transformation  $T$  on a vector space  $V(F)$  to be invertible.
16. Define characteristic value.

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17. Define characteristic vector.
18. Distinct characteristic vectors of  $T$  corresponding to distinct characteristic value of  $T$  are linearly independent.  
( Write True or False )
19. Write down the characteristic equation of the matrix  
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
20. State Cayley-Hamilton theorem.
21. Define norm of a vector in inner product space.
22. Define unit vector of a vector in inner product space.
23. Define orthogonality.
24. State Cauchy-Schwarz inequality.
25. State Bessel's inequality.

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SECTION—B

Answer any five of the following questions :  $2 \times 5 = 10$

26. Let  $V(F)$  be a vector space and  $0$  be the zero vector of  $V$ . Then show that  
 $\alpha v = 0 \Rightarrow \alpha = 0$  or  $v = 0, \forall \alpha \in F$  and  $\forall v \in V$
27. Show that the vectors  $(1, 1, 0, 0), (0, 1, -1, 0), (0, 0, 0, 3)$  in  $R^4$  are linearly independent.
28. Let  $T: U \rightarrow V$  be a linear transformation. Then prove that  
 $T(-u) = -T(u), \forall u \in U$
29. The mapping  $T: V_2(R) \rightarrow V_3(R)$  defined as  
 $T(a, b) = (a + b, a - b, b)$   
is a linear transformation. Find the null space of  $T$ .
30. If  $T: U \rightarrow V$  is a homomorphism, then prove that  $T(0) = 0'$  where  $0$  and  $0'$  are the zero vectors of  $U$  and  $V$  respectively.
31. Define isomorphism of a vector space.
32. Let  $T$  be a linear operator on a finite-dimensional vector space  $V$  and let  $\lambda$  be the characteristic value of  $T$ . Then prove that the set  $W_\lambda = \{v \in V | Tv = \lambda v\}$  is a subspace of  $V$ .

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33. If  $\lambda$  is a characteristic value of an invertible transformation  $T$ , then show that  $\lambda^{-1}$  is a characteristic value of  $T^{-1}$ .

34. In an inner product space  $V(F)$ , prove that  $\|\alpha x\| = |\alpha| \|x\|$

35. Define orthonormal set in inner product space.

SECTION—C

Answer any five of the following questions :  $8 \times 5 = 40$

36. (a) Prove that the necessary and sufficient condition for a non-empty subset  $W$  of a vector space  $V(F)$  to be a subspace of  $V$  is  $\alpha x + \beta y \in W, \forall \alpha, \beta \in F$  and  $\forall x, y \in W$ . 5

(b) Show that the vectors  $(1, 2, 1), (2, 1, 0), (1, -1, 2)$  form a basis of  $R^3$ . 3

37. In any finite-dimensional vector space, prove that any two bases have the same number of elements. 8

38. (a) The function  $T: V_3(R) \rightarrow V_2(R)$  defined by  $T(a, b, c) = (a, b), \forall a, b, c \in R$ . Then show that  $T$  is a linear transformation from  $V_3(R)$  into  $V_2(R)$ . 3

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(b) If  $U(F)$  and  $V(F)$  are two vector spaces and  $T: U \rightarrow V$  is a linear transformation, then show that range of  $T$  is a subspace of  $V$  and kernel of  $T$  is a subspace of  $U$ . 5

39. (a) Show that the mapping  $T: V_2(R) \rightarrow V_3(R)$  defined as  $T(a, b) = (a+b, a-b, b)$  is a linear transformation from  $V_2(R)$  into  $V_3(R)$ . 3

(b) Let  $T$  be the linear operator on  $R^2$  defined by

$$T(x, y) = (4x - 2y, 2x + y)$$

Compute the matrix of  $T$  relative to the basis  $\{\alpha_1, \alpha_2\}$ , where  $\alpha_1 = (1, 1)$  and  $\alpha_2 = (-1, 0)$ . 5

40. Prove that two finite-dimensional vector spaces over the same field are isomorphic if and only if they are of the same dimension. 8

41. If  $T: U \rightarrow V$  is an isomorphism of the vector space  $U$  into  $V$ , then prove that a set of vectors  $\{T(u_1), T(u_2), \dots, T(u_r)\}$  is linearly independent if and only if the set  $\{u_1, u_2, \dots, u_r\}$  is linearly independent. Give example to show that the same does not hold if  $T$  is not isomorphism.  $6+2=8$





42. Let  $T$  be a linear operator on a finite-dimensional vector space  $V$ . Then prove that the following are equivalent : 8
- (a)  $\lambda$  is a characteristic value of  $T$
  - (b) The operator  $T - \lambda I$  is singular
  - (c)  $|T - \lambda I| = 0$

43. Find all eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
8

44. (a) If  $x, y$  are vectors in an inner product space  $V$ , then prove that

$$\|x + y\| \leq \|x\| + \|y\|$$
5

- (b) If in an inner product space the vectors  $x$  and  $y$  are linearly dependent, then show that  $|(x, y)| = \|x\| \|y\|$ . 3

45. Prove that every finite-dimensional inner product space has an orthonormal basis. 8

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