2022/TDC (CBCS)/EVEN/SEM/ MTMHCC-602T/265

TDC (CBCS) Even Semester Exam., 2022

MATHEMATICS

(Honours)

(6th Semester)

Course No. : MTMHCC-602T

(Linear Algebra)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following questions : 2×10=20

1. Let $V(F)$ be a vector space and 0 be the zero vector of V. Then show that

$$\alpha v = 0 \Rightarrow \alpha = 0 \quad \text{or} \quad v = 0,\ \forall\ \alpha \in F \text{ and } \forall\ v \in V$$

2. Is the set

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\} \subseteq \mathbb{R}^3$$

linearly independent over $\mathbb{R}$? Justify your answer.

3. Define basis of a vector space and give an example.

4. Is $T: R^2 \rightarrow R^3$ defined as

$$T(x, y) = (x+3, 2y, x+y)$$

a linear transformation? Justify your answer.

5. State Sylvester's law of nullity.

6. The mapping $T: V_2(R) \rightarrow V_3(R)$ defined as

$$T(a, b) = (a+b, a-b, b)$$

is a linear transformation. Find the null space of T.

7. If $T: U \rightarrow V$ is a homomorphism, then prove that $T(0) = 0'$ where 0 and $0'$ are the zero vectors of U and V respectively.

8. Define homomorphism of a vector space and give an example.

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9. Is the linear transformation $T: R^2 \to R^2$ defined by

$$T(x, y) = (x + 2y, 2x + 4y) \quad \forall (x, y) \in R^2$$

an isomorphism? Justify your answer.

10. Let T be an invertible linear operator on a finite-dimensional vector space V over a field F. Prove that $\lambda \in F$ is a characteristic root of T if and only if λ^{-1} is a characteristic root of T^{-1}.

11. Show that the eigenvalues of a diagonal matrix are its diagonal elements.

12. Prove that eigenvalues of unitary matrix are of unit modulus.

13. In an inner product space $V(F)$, prove that

$$\|\alpha x\| = |\alpha| \, \|x\|$$

14. When are two vectors said to be orthonormal in an inner product space?

15. Prove that an orthonormal set of vectors in an inner product space V is linearly independent.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. *(a)* Prove that union of two subspaces is a subspace if and only if one of them contains the other. 5

(b) Show that every linearly independent subset of a finitely generated vector space is a basis or can be extended to form a basis. 5

17. *(a)* If W_1 and W_2 are two subspaces of a finite-dimensional vector space $V(F)$, then prove that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

5

(b) If W_1 and W_2 be two subspaces of a vector space $V(F)$, then show that

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1,\ w_2 \in W_2\}$$

is a subspace of $V(F)$ spanned by $W_1 \cup W_2$. 5

18. *(a)* If V is a vector space and $T : V \to V$ is a linear operator, then prove that the following are equivalent : 5

(i) $\text{Range}(T) \cap \ker(T) = \{0\}$

(ii) $T(Tx) = 0 \Rightarrow Tx = 0$

(b) Let T be the linear operator on R^2 defined by $T(x, y) = (4x - 2y, 2x + y)$. Compute the matrix T relative to the basis $B = \{(1, 1), (-1, 0)\}$. 5

9. (a) Prove that there exists a linear transformation $T : R^2 \to R^3$ such that

$$T(1, 1) = (1, 0, 2) \text{ and } T(2, 3) = (1, -1, 4)$$

What is $T(8, 11)$? 4+1=5

(b) Find range, rank, kernel and nullity of the linear transformation $T : R^2 \to R^2$ defined by $T(x, y) = (x + y, x)$. 5

10. (a) Prove that two finite-dimensional vector spaces over the same field are isomorphic if and only if they are of the same dimension. 7

(b) Prove that isomorphism is an equivalence relation. 3

11. (a) If $T : U \to V$ is an isomorphism of the vector space U into V, then prove that the set of vectors $\{T(u_1), T(u_2), \cdots, T(u_r)\}$ is linearly independent if and only if the set $\{u_1, u_2, \cdots, u_r\}$ is linearly independent. Give example to show that the same does not hold if T is not isomorphism. 6+2=8

(b) Define isomorphism of a vector space and give an example. 2

22. (a) Let T be a linear operator on V, where V is a vector space over a field F. If $v_1, v_2, \cdots, v_n$ are non-zero eigenvectors of T belonging to distinct eigen-values $\lambda_1, \lambda_2, \cdots, \lambda_n$, then show that $v_1, v_2, \cdots, v_n$ are linearly independent. 5

(b) Show that eigenvalues of

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

are ± 1 and the corresponding eigen-vectors are

$$\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \text{ and } \begin{pmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{pmatrix}$$ 5

23. (a) Let T be a linear operator on a finite-dimensional vector space V. Then prove that the following are equivalent : 7

(i) λ is a characteristic value of T

(ii) The operator $T - \lambda I$ is singular

(iii) $|T - \lambda I| = 0$

(b) Prove that similar matrices have the same characteristic polynomial. 3

24. (a) If x, y are vectors in an inner product space V, then prove that 5

$$||x + y|| \leq ||x|| + ||y||$$

(b) Prove that if V is an inner product space, then $|\langle x, y\rangle| = \|x\| \|y\|$ if and only if one of x or y is a multiple of the other. 5

5. (a) State and prove Cauchy-Schwartz inequality in an inner product space. 1+5=6

(b) If $\{v_1, v_2, \cdots, v_n\}$ is an orthonormal subset of an inner product space V, then prove that for any $v \in V$, the vector

$$v - \sum_{i=1}^{n} (v, v_i) v_i$$

is perpendicular to each v_i. 4

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