2022/TDC (CBCS)/EVEN/SEM/
MTMHCC-601T/264

TDC (CBCS) Even Semester Exam., 2022

MATHEMATICS

(Honours)

(6th Semester)

Course No. : MTMHCC-601T

(Complex Analysis)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

wer any *ten* of the following questions : 2×10=20

If $|z|=1$, then find the real part of

$$\frac{z-1}{z+1}$$

Find arg $i(x+iy)$, if $\arg(x+iy)=\alpha$.

3. Show that for any two complex numbers z and w, $|z-w| \geq |z|-|w|$.

4. Justify if the function $f: \mathbf{C} \to \mathbf{C}$ defined by $f(z) = \bar{z}$ is differentiable at $z = 0$.

5. Check if Cauchy-Riemann equations are satisfied for $f(z) = |z|^2$ at $z = 1 + i$.

6. Show that if $f(z)$ is analytic at z_0, then it must be continuous at z_0.

7. Explain simply connected region and multiply connected region.

8. Evaluate $\int (2y + x^2)\,dx + (3x - y)\,dy$ along the arc of the parabola $x = 2t$, $y = t^2 + 3$ joining $(0, 3)$ to $(2, 4)$.

9. If C is any simple closed curve, evaluate

$$\oint_C z\,dz$$

10. Justify if $\sin z$ is an entire function.

11. State fundamental theorem of algebra. How many roots does the equation $z^{100} - 1 = 0$ have?

12. Use $(\varepsilon\text{-}\delta)$ definition of limit to show that

$$\underset{n \to \infty}{\text{Lt}} \left(1 + \frac{z}{n}\right) = 1$$

for each $z \in \mathbf{C}$.

13. Give a brief description of Laurent series of a complex function about a singular point.

14. Find Laurent series for the function

$$f(z) = \frac{z - \sin z}{z^3}$$

about $z = 0$.

15. Let

$$f(z) = \frac{z}{(z-1)(z+1)^2}$$

Compute the residues at all the poles of this function.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. (a) Explain the geometrical interpretation of

$$\arg\left(\frac{z-\alpha}{z-\beta}\right)$$

Hence find the condition for collinearity of three complex numbers z, α and β. 4+1=5

(b) If z_1, z_2, z_3 are the vertices of an isosceles triangle, then show that

$$z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$$

Hence show that

$$(z_1 + z_3)^2 = 2(z_1 - z_2)(z_2 - z_3)$$

4+1=5

17. (a) Determine the region of the complex plane described by $|z+1|+|z-1| \leq 4$. Illustrate the same with a diagram. 4+1=5

(b) Define limit of a complex function at a point. Show that

$$\operatorname*{Lt}_{z \to z_0} f(z)$$

if it exists, is unique. 1+4=5

(a) Prove that the function $|z|^2$ is continuous everywhere but nowhere differentiable except origin. 5

(b) Show that the function

$$u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

satisfies Laplace equation and determine the corresponding analytic function of which it is the real part. 2+3=5

19. (a) If $f(z) = u(x, y) + iv(x, y)$ is analytic in $D \subseteq \mathbb{C}$, then show that u and v satisfy the Cauchy-Riemann equations in D. 5

(b) Derive the polar form of Cauchy-Riemann equations. 5

20. (a) Prove Cauchy-Goursat theorem for a triangle. 6

(b) Evaluate

$$\oint_C \frac{dz}{z-3}$$

where C is the circle $|z-2| = 5$. Does the result contradict Cauchy's theorem? Justify. 3+1=4

21. *(a)* Prove Cauchy's integral formula.

(b) Use Cauchy's integral formula to evaluate

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$$

where C is the circle $|z| = 3$.

22. *(a)* State and prove Liouville's theorem. 1+4

(b) Prove that every polynomial equation

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$$

$n \geq 1$ and $a_n \neq 0$ has exactly n roots.

23. *(a)* Prove that an absolutely convergent series is convergent. Is the converse true? Justify. 4+1=

(b) Prove that the series

$$z(1-z) + z^2(1-z) + z^3(1-z) + \cdots$$

converges for $|z| < 1$ and find its sum.

4+1=

24. *(a)* Expand $\dfrac{e^z}{z^3}$ and $\dfrac{z^3}{e^z}$ in Laurent series about $z = 0$. Hence identify the types of singularities in each case. 3+2=

(b) Evaluate :

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta$$ 5

(a) Evaluate

$$\oint_C \frac{2+3\sin \pi z}{z(z-1)^2}\, dz$$

where C is the square with vertices $3+3i$, $3-3i$, $-3+3i$ and $-3-3i$. 5

(b) Given $a > |b|$, show that

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$ 5

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